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ASYMPTOTIC BEHAVIOR OF WRONSKIAN OF BOUNDARY CONDITION FUNCTIONS FOR A SECOND ORDER BOUNDARY VALUE PROBLEM

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Abstract: The Wronskian of boundary condition functions for a second order boundary value problem is shown to be asymptotically equivalent to the Wronskian of the corresponding Fourier problem.

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1. Introduction

Although research on boundary condition functions has not been very prolific lately, E.C. Titchmarsh (see [7], [8]) did some exhaustive work in this field. W.N. Everitt (see [1], [2], [3]) and D.N. Offei (see [4], [5]) have also looked at some aspects of boundary condition functions. In [6], the boundary condition functions for the second order boundary value problem

$$L\phi \equiv -\phi^(2)(x) + p(x)\phi(x) = \lambda\phi(x) \quad (a \leq x \leq b), \tag{1}$$

$$U_r\phi \equiv \sum_{s=1}^{2}[\alpha_{rs}\phi^{(s-1)}(a) + \beta_{rs}\phi^{(s-1)}(b) = 0 \quad (1 \leq r \leq 2), \tag{2}$$

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where the function p, the boundary constants αrs, βrs and the parameter λ are complex with separated boundary conditions or otherwise shown to be asymptotically equivalent for large values of $|\lambda|$ to the boundary condition functions of the corresponding Fourier problem

$$L\phi \equiv -\phi^{(2)}(x) = \lambda\phi(x), \tag{3}$$

$$U_r\phi \equiv \sum_{s=1}^{2}[\alpha_{rs}\phi^{(s-1)}(a) + \beta_{rs}\phi^{(s-1)}(b)] = 0 \quad (1 \le r \le 2) \tag{4}$$

In this paper, we show that the Wronskian of the boundary condition functions of equations (1)-(2) are asymptotically equivalent to the Wronskian and Green's function of the corresponding Fourier problem (3)-(4).

Again in [6], we showed that if $\psi_r(a|x,\lambda)$ and $\chi_r(b|x,\lambda)$, $(1 \le r \le 2)$ are boundary condition functions for (1)-(2) and $\psi_{Fr}(a|x,\lambda)$ and $\chi_{Fr}(b|x,\lambda)$,$(1 \le r \le 2)$ are boundary condition functions for (3)-(4) then

1. $\psi_r(a|x,\lambda) \sim \psi_{Fr}(a|x,\lambda)$;
2. $\chi_r(a|x,\lambda) \sim \chi_{Fr}(a|x,\lambda)$

as $|\lambda|$, $(1 \le r \le 2)$. Now if,

$$\eta_r(x,\lambda) = \psi_r(a|x,\lambda) + \chi_r(a|x,\lambda),$$

$$\eta_{Fr}(x,\lambda) = \psi_{Fr}(a|x,\lambda) + \chi_{Fr}(a|x,\lambda),$$

and

$$W(\lambda) = W(\eta_1(x,\lambda), \eta_2(x,\lambda))(x),$$
$$W_F(\lambda) = W(\eta_{F1}(x,\lambda), \eta_{F2}(x,\lambda))(x),$$

then the zeros of $W(\lambda)$ and $W_F(\lambda)$ are the eigenvalues of (1) and (3) respectively. In this paper, we show that

$$W(\lambda) \sim W_F(\lambda)$$

for suitably large values of $|\lambda|$, that is avoiding the eigenvalues where $W_F(\lambda)$ has zeros.

2. Preliminaries

In this section we shall give some notation and properties of the linear differential operator L defined by

$$L\phi = P_0(x)\phi^(2)(x) + P_1(x)\phi'(x) + P_2(x)\phi(x). \tag{5}$$

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For a suitable pair of functions $\phi_1(x)$ and $\phi_2(x)$, the symbol $\Phi(x)$ denotes the 2×2 matrix $\left[\phi_r^{(s-1)}(x)\right]$ $(1 \le r, s \le 2)$ so that

$$\Phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix}$$

and

$$W(\phi_1, \phi_2)(x) = det(\Phi)(x).$$

Closely associated with L is another differentail expression L^+ called the Lagrange adjoint of L and given by

$$L^+\psi = \overline{P_0}(x)\psi^{(2)}(x) + \left[2\overline{P_0}(x) - \overline{P_1}(x)\right]\psi'(x) + \overline{P_0}^{(2)}(x) - \overline{P_1}'(x) + \overline{P_2}(x)\psi(x). \quad (6)$$

For a suitable pair of functions f and g

$$\int_a^b \left(\overline{g}Lf - f\overline{L}^+\right) dx = [fg](b) - [fg](a). \quad (7)$$

Here $[fg](x)$ is the bilinear form in (f, f') and $(\overline{g}, \overline{g}')$ given by

$$[fg] = \sum_{j=1}^{2}\sum_{k=1}^{2} B_{jk}(x)\overline{g}^{(j-1)}(x)f^{(k-1)}(x) \quad (8)$$

$$= \hat{g}^*(x)B(x)\hat{f}(x), \quad (9)$$

where $\hat{f}(x)$ denotes the column vector with components $f(x)$, $f'(x)$ and $\hat{g}^*(x)$ denotes the row vector with components $\overline{g}(x), \overline{g}'(x)$. The matrix $B(x)$ is given by

$$B(x) = \begin{bmatrix} P_1(x) - P_0(x) & P_0(x) \\ -P_0(x) & 0 \end{bmatrix}. \quad (10)$$

The notation A^* is used to represent the conjugate transpose of the matrix A. If there exists $k > 0$ such that $|f(x)| \le k\phi(x)$ for some $x \ge x_0$, then we write

$$f = O(\phi) \text{ as } x \to \infty. \quad (11)$$

Also if $g = O(\psi)$ as $x \to \infty$, then

$$\left.\begin{array}{rcl} f + g &=& O(\phi + \psi), \\ f.g &=& O(\phi.\psi), \\ k.g &=& O(\phi), \end{array}\right\} \quad (12)$$

where k is a constant. Furthermore, $f \sim l\phi$ as $x \to \infty$, where $l \ne 0$ means that $\frac{f}{\phi} \to l$ as $x \to \infty$.

If $\phi(x,\lambda)$ is the solution of $L\phi = \lambda\phi$ and $\psi(x,\lambda)$ is the solution of $L^{+}\psi = \overline{\lambda}\psi$ then

$$[\phi\psi](x_2) - [\phi\psi](x_1) = \int_{x_1}^{x_2} \{\psi L\phi - \phi\overline{L}^{+}\}dx = 0 \quad a \le x_1 \le x_2 \le b. \tag{13}$$

Thus $[\phi\psi]$ is independent of $x \in [a,b]$. If $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are solutions $\phi(x,\lambda)$ and if $x_1, x_2 \in [a,b]$ then $W(\phi_1,\phi_2)(x_1) = W(\phi_1,\phi_2)(x_2)$ so $W(\phi_1,\phi_2)(x)$ is independent of $x \in [a,b]$. For the special case where $P_0(x) = -1$ and $P_2(x) = 0$

$$[fg](x) = W(f,\overline{g})(x). \tag{14}$$

3. Determination of Eigenvalues for the Boundary Value Problem

Let $\{\phi_1(x,\lambda), \phi_2(x,\lambda)\}$ be a fundamental set of solutions for (1). Then every solution of (1) is of the form

$$\phi(x,\lambda) = A_1\phi_1(x,\lambda) + A_2\phi_2(x,\lambda), \tag{15}$$

where A_1 and A_2 are constants. The function $\phi(x,\lambda)$ is an eigenfunction of equations (1)-(2) if and only if A_1 and A_2 are not both zero and $\phi(x,\lambda)$ satisfies equation (2) that is,

$$\begin{aligned} U_r\phi(x,\lambda) &= U_r(A_1\phi_1(x,\lambda) + A_2\phi_2(x,\lambda)) \\ &= U_r(A_1\phi_1(x,\lambda)) + U_r(A_2\phi_2(x,\lambda)) \\ &= A_1U_r\phi_1(x,\lambda) + A_2U_r\phi_2(x,\lambda) = 0 \quad (1 \le r \le 2). \end{aligned} \tag{16}$$

Since A_1 and A_2 are not both zero then from [6] we have

$$\begin{vmatrix} U_1\phi_1 & U_1\phi_2 \\ U_2\phi_1 & U_2\phi_2 \end{vmatrix} = \begin{vmatrix} W(\phi_1,\overline{\eta}_1) & W(\phi_1,\overline{\eta}_2) \\ W(\phi_2,\overline{\eta}_1) & W(\phi_2,\overline{\eta}_2 \end{vmatrix} = 0. \tag{17}$$

This is the equation for λ and the values of λ which satisfy it are the eigenvalues of the boundary value problem (1)-(2). Therefore from (17) , λ is the eigenvalue of (1)-(2) if and only if

$$\begin{vmatrix} W(\phi_1,\overline{\eta}_1) & W(\phi_1,\overline{\eta}_2) \\ W(\phi_2,\overline{\eta}_1) & W(\phi_2,\overline{\eta}_2 \end{vmatrix} = W(\phi_1,\phi_2)(x)W(\overline{\eta}_1,\overline{\eta}_2)(x) = 0. \tag{18}$$

Since η_1 and η_2 are solutions of $L^{+}\psi = \overline{\lambda}\psi$, then $W(\eta_1,\eta_2)(x,\lambda)$ is also independent of $x \in [a,b]$.

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Theorem 3.1. *Let*

$$W(\lambda) = W(\eta_1, \eta_2)(x, \lambda) \quad x \in [a, b].$$

Then λ *is an eigenvalue of the boundary value problem* (1)-(2) *if and only if* $W(\lambda) = 0$.

Proof. Suppose λ is an eigenvalue of (1)-(2). Then from (18), we have

$$W(\overline{\eta}_1, \overline{\eta}_2)(x, \lambda) = 0$$

since $\{\phi_1, \phi_2\}$ so $W(\phi_1, \phi_2) \neq 0$ for $x \in [a, b]$.

This means that $\overline{\eta}_1, \overline{\eta}_2$ are linearly independent on $[a, b]$. Consequently, η_1, η_2 are linearly independent on $[a, b]$ and hence

$$W(\lambda) = W(\eta_1, \eta_2)(x, \lambda) = 0.$$

Conversely, suppose

$$W(\eta_1, \eta_2)(x, \lambda) = 0.$$

Then $W(\overline{\eta}_1, \overline{\eta}_2)(x, \lambda) = 0$ and therefore (18) holds so that λ is an eigenvalue. □

From Theorem 3.1, it is easy to show that if $w(\lambda) = W(\zeta_1, \zeta_2)$ then λ is an eigenvalue of the adjoint boundary value problem for (1)-(2) if and only if $w(\lambda) = 0$.

4. Asymptotic Behavior of Wronskian of Boundary Condition Functions

In [6], we showed that if

$$\psi_r(a|x, \lambda), \chi_r(b|x, \lambda) \quad (1 \le r \le 2)$$

are the boundary condition functions for (1)-(2) and

$$\psi_{Fr}(a|x, \lambda), \chi_{Fr}(b|x, \lambda) \quad (1 \le r \le 2)$$

are the boundary condition functions for (3)-(4), then

$$\psi_r^{(s-1)} \sim \psi_{Fr}^{(s-1)} \quad (1 \le r, s \le 2) \quad \text{as } |\lambda| \to \infty,$$

$$\chi_r^{(s-1)} \sim \chi_{Fr}^{(s-1)} \quad (1 \le r, s \le 2) \quad \text{as } |\lambda| \to \infty.$$

Now, let

$$\left.\begin{array}{rcl} \eta_r(x) &=& \psi_r(a|x, \lambda) + \chi_r(b|x, \lambda), \\ \eta_{Fr}(x) &=& \psi_{Fr}(a|x, \lambda) + \chi_{Fr}(b|x, \lambda), \\ W(\lambda) &=& W(\eta_1, \eta_2)(x), \\ W_F(\lambda) &=& W(\eta_{F1}, \eta_{F2})(x), \end{array}\right\} \tag{19}$$

then

$$W_F(x) = \begin{vmatrix} \psi_{F1}(a|x,\lambda) + \chi_{F1}(b|x,\lambda) & \psi_{F2}(a|x,\lambda) + \chi_{F2}(b|x,\lambda) \\ \psi'_{F1}(a|x,\lambda) + \chi'_{F1}(b|x,\lambda) & \psi'_{F2}(a|x,\lambda) + \chi'_{F2}(b|x,\lambda) \end{vmatrix}$$
$$= W(\psi_{F1},\psi_{F2})(x) + W(\psi_{F1},\chi_{F2})(x) + W(\chi_{F1},\psi_{F2})(x) + W(\chi_{F1},\chi_{F2})(x). \quad (20)$$

Similarly,

$$W(\lambda) = W(\psi_1,\psi_2)(x) + W(\psi_1,\chi_2)(x) + W(\chi_1,\psi_2)(x) + W(\chi_1,\chi_2)(x). \quad (21)$$

Each of the determinants is independent of $x \in [a,b]$ and so we set $x = a$ if all the determinants have all ψ_r or $\psi_{Fr}(1 \le r \le 2)$ and $x = b$ when determinant includes all χ_r or $\chi_{Fr}(1 \le r \le 2)$. We can either set $x = a$ or $x = b$ in the other 2 determinants.

In [6], we showed that

$$\begin{bmatrix} \psi_1(a) & \psi_2(a) \\ \psi'_1(a) & \psi'_2(a) \end{bmatrix} = \begin{bmatrix} -\overline{\alpha}_{12} & -\overline{\alpha}_{22} \\ \overline{\alpha}_{11} & \overline{\alpha}_{21} \end{bmatrix}, \quad (22)$$

$$\begin{bmatrix} \chi_1(a) & \chi_2(a) \\ \chi'_1(a) & \chi'_2(a) \end{bmatrix} = \begin{bmatrix} -\overline{\beta}_{12} & -\overline{\beta}_{22} \\ \overline{\beta}_{11} & \overline{\beta}_{21} \end{bmatrix}, \quad (23)$$

$$\begin{bmatrix} \psi_{F1}(a) & \psi_{F2}(a) \\ \psi'_{F1}(a) & \psi'_{F2}(a) \end{bmatrix} = \begin{bmatrix} -\overline{\alpha}_{12} & -\overline{\alpha}_{22} \\ \overline{\alpha}_{11} & \overline{\alpha}_{21} \end{bmatrix}, \quad (24)$$

$$\begin{bmatrix} \chi_{F1}(a) & \chi_{F2}(a) \\ \chi'_{F1}(a) & \chi'_{F2}(a) \end{bmatrix} = \begin{bmatrix} -\overline{\beta}_{12} & -\overline{\beta}_{22} \\ \overline{\beta}_{11} & \overline{\beta}_{21} \end{bmatrix}, \quad (25)$$

so that

$$\Psi(a) = \Psi_F(a), \quad \chi(b) = \chi_F(b). \quad (26)$$

Let

$$W(\psi_{F1},\psi_{F2})(a) = \alpha, \quad W(\chi_{F1},\chi_{F2})(b) = \beta, \quad (27)$$

then (20) can be written as

$$W_F(\lambda) = \alpha + \beta + W(\psi_{F1},\chi_{F2})(b) + W(\chi_{F1},\psi_{F2})(a). \quad (28)$$

Similarly, (21) can be written as

$$W(\lambda) = \alpha + \beta + W(\psi,\chi)(b) + W(\chi,\psi)(a). \quad (29)$$

Theorem 4.1. *$W_F(\lambda) = 0(|\omega|e^{|\tau|(b-a)})$ as $|\lambda| \to \infty$.*

Proof. From (25),

$$\begin{aligned} W(\psi_{F1},\chi_{F2})(b) &= \begin{vmatrix} \psi_{F1}(b) & \chi_{F2}(b) \\ \psi'_{F1}(b) & \chi'_{F2}(b) \end{vmatrix} \\ &= \begin{vmatrix} \psi_{F1}(b) & -\overline{\beta}_{22} \\ \psi'_{F1}(b) & \overline{\beta}_{21} \end{vmatrix} \\ &= \overline{\beta}_{21}\psi_{F1}(b) + \overline{\beta}_{22}\psi'_{F1}(b). \end{aligned} \tag{30}$$

In [6], we showed that

$$\psi_{F1}(b) = 0(e^{|\tau|(b-a)}), \quad \psi'_{F1}(b) = 0(|\omega|\, e^{|\tau|(b-a)}). \tag{31}$$

From (12) and (31) we have

$$W(\psi_{F1},\chi_{F2})(b) = 0(|\omega|\, e^{|\tau|(b-a)}). \tag{32}$$

From (24)

$$\begin{aligned} W(\chi_{F1},\psi_{F2})(a) &= \begin{vmatrix} \chi_{F1}(a) & \psi_{F2}(a) \\ \chi'_{F1}(a) & \psi'_{F2}(a) \end{vmatrix} \\ &= \begin{vmatrix} \chi_{F1}(b) & -\overline{\alpha}_{22} \\ \chi'_{F1}(b) & \overline{\alpha}_{21} \end{vmatrix} \\ &= \overline{\alpha}_{21}\chi_{F1}(a) + \overline{\alpha}_{22}\chi'_{F1}(a). \end{aligned} \tag{33}$$

In [6] we showed that

$$\chi_{F1}(a) = 0(e^{|\tau|(b-a)})\chi'_{F1}(a) = 0(|\omega|\, e^{|\tau|(b-a)}). \tag{34}$$

From (12) and (34), (33) reduces to

$$W(\chi_{F1},\psi_{F2})(a) = 0(|\omega|\, e^{|\tau|(b-a)}). \tag{35}$$

From (32) and (35), (28) becomes

$$W_F(\lambda) = 0(|\omega|\, e^{|\tau|(b-a)}) \ \text{ as } \ |\lambda| \to \infty. \tag{36}$$

□

Theorem 4.2.

$$W(\psi_1,\chi_2)(b) = W(\psi_{F1},\chi_{F2})(b) + 0(e^{|\tau|(b-a)}),$$

$$W(\chi_1,\psi_2)(a) = W(\chi_{F1},\psi_{F2})(a) + 0(|\omega| e^{|\tau|(b-a)})$$

as $|\lambda| \to \infty$.

Proof. From equation (23)

$$\begin{aligned} W(\psi_1,\chi_2)(b) &= \begin{vmatrix} \psi_1(b) & \chi_2(b) \\ \psi_1'(b) & \chi_2'(b) \end{vmatrix} \\ &= \begin{vmatrix} \psi_1(b) & -\overline{\beta}_{22} \\ \psi_1'(b) & \overline{\beta}_{21} \end{vmatrix} \\ &= \overline{\beta}_{21}\psi_1(b) + \overline{\beta}_{22}\psi_1'(b). \end{aligned} \tag{37}$$

In [6] we showed that

$$\psi_1(b) = \psi_{F1}(b) + 0(|\omega|^{-1}\, e^{|\tau|(b-a)}), \quad \psi_1(b) = \psi_{F1}'(b) + 0(e^{|\tau|(b-a)}). \tag{38}$$

From (12) and (38) we can rewrite (37) as

$$W(\psi_1,\chi_2)(b) = W(\psi_{F1},\chi_{F2})(b) + 0(|\omega|\, e^{|\tau|(b-a)}). \tag{39}$$

From (22)

$$\begin{aligned} W(\chi_1,\psi_2)(a) &= \begin{vmatrix} \chi_1(a) & \psi_2(a) \\ \chi_1'(a) & \psi_2'(a) \end{vmatrix} \\ &= \begin{vmatrix} \chi_1(a) & -\overline{\alpha}_{22} \\ \chi_F 1'(a) & \overline{\alpha}_{21} \end{vmatrix} \\ &= \overline{\alpha}_{21}\chi_F 1(a) + \overline{\alpha}_{22}\chi_1'(a). \end{aligned} \tag{40}$$

In [6], we showed that

$$\chi_1(a) = \chi_{F1}(a) + 0(|\omega|^{-1}\, e^{|\tau|(b-a)}), \quad \chi_1'(a) = \chi_{F1}'(a) + 0(|\omega|\, e^{|\tau|(b-a)}). \tag{41}$$

From (12) and (41), (40) reduces to

$$W(\chi_1,\psi_2)(a) = W(\chi_{F1},\psi_{F2})(a) + 0(e^{|\tau|(b-a)}). \tag{42}$$

□

Theorem 4.3. *$W(\lambda) \sim W_F(\lambda)$ for suitably large values of $|\lambda|$ that is, avoiding the eigenvalues where $W_F(\lambda) = 0$.*

Proof. From (29) and Theorem 4.2,

$$W(\lambda) = \alpha + \beta + W(\psi_{F1},\chi_{F2})(b) + W(\chi_{F1},\psi_{F2})(a) + 0(e^{|\tau|(b-a)}). \tag{43}$$

So from (28)

$$W(\lambda) = W_F(\lambda) + 0(e^{|\tau|(b-a)}). \tag{44}$$

From Theorem 4.1 and (44) we have the result. □

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