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NEW INTEGRAL TRANSFORM USING CURZON'S INTEGRAL

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Abstract: Using Curzon's integral which gives relation between Hermite polynomials and Legendre polynomials [5] and integral transform [6] we develop a new pair of integral transform and briefly give its use to solve the differential equation of the type $L_{op}g(z) = h(z)$, where L_{op} is a linear differential operator.

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1. Introduction

Curzon [5] obtained many relations between Hermite polynomials and Legendre polynomials, i.e. between $H_n(t)$ and $P_n(t)$ with n usually not restricted to be integral. One of the simplest of his relations, in which n is an integer, is

$$P_n(t) = \frac{2}{n!\sqrt{\pi}} \int_0^\infty \exp(-z^2) z^n H_n(tz) dz. \tag{1}$$

Here, the Hermite polynomials $H_n(t)$ is defined as,

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!},$$

valid for all finite x and t and the Legendre polynomials $P_n(t)$ is defined as,

$$g(z) = \sum c_n j_n(z) \left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Here $(1 - 2xt + t^2)^{-1/2}$ denotes the particular branch which $\to 1$ as $t \to 0$. Using

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Bauer's expansion and properties of spherical Bessel and Legender functions, B.G. Sidharth [6] derived

$$\int_{-1}^{+1} e^{izt} P_n(t)dt = 2i^n J_n(z) \tag{2}$$

and

$$\int_{-\infty}^{\infty} j_n(z)e^{izt}dz = 2i^n P_n(t). \tag{3}$$

We consider that z is real and also note that,

$$j_n(-z) = (-1)^n j_n(z). \tag{4}$$

2. Integral Transform

Let us consider a function $g(z)$ which can be expanded as an infinite linear combination spherical Bessel functions, on the lines of Neumann's expansion in terms of ordinary Bessel functions. This can be done because of the orthogonality relations. Similarly we will also use the known expansion in terms of Legender functions. Thus we have,

$$g(z) = \sum c_n j_n(z),$$

by (2) we get

$$g(z) = \sum c_n (2i^n)^{-1} \int_{-1}^{+1} e^{izt} P_n(t)dt,$$

or

$$g(z) = \int_{-1}^{+1} f(t)e^{izt}dt, \tag{5}$$

where

$$f(t) = \sum k_n P_n(t) \qquad \left(k_n = c_n(2i^n)^{-1}\right).$$

Using (1) we get

$$\begin{aligned} f(t) &= \int_0^{\infty} \sum \frac{c_n}{2i^n} \frac{2}{n!\sqrt{\pi}} \exp(-z^2)z^n H_n(tz)dz \\ &= K \int_0^{\infty} \exp(-z^2)z^n H_n(tz)dz, \end{aligned}$$

where

$$K = \sum \frac{c_n}{n!\sqrt{\pi}}.$$

Putting $z = x$, we obtain

$$f(t) = K \int_0^{\infty} g(x)H_n(tx)dx, \tag{6}$$

where $g(x) = \exp(-x^2)x^n$.

Finally by using (5) we get

$$g(z) = K \int_{-1}^{+1} \int_{0}^{\infty} g(x) H_n(tx) e^{izt} dx dt. \tag{7}$$

The relations (5), (6) and (7) are the desired developed integral transform relations.

3. Application

Consider the differential equation,

$$L_{op} g(z) = h(z), \tag{8}$$

where L_{op} is a linear differential operator. To solve this differential equation means to find the value of $g(z)$. This can be done as follows:

Using (7) in (8), we get

$$L_{op} g(z) = F\left(\frac{d}{dz}\right) g(z) = K \int_{-1}^{+1} \int_{0}^{\infty} F(it) g(x) H_n(tx) e^{izt} dx dt = h(z).$$

Now using (6):

$$L_{op} g(z) = A \int_{-1}^{+1} f(t) F(it) e^{izt} dt = h(z).$$

Hence

$$\begin{aligned} h(z) &= A \int_{-1}^{+1} f(t) F(it) e^{izt} dt \\ &= \int_{-1}^{+1} \hat{h}(t) e^{izt} dt. \end{aligned}$$

Here $\hat{h}(t) = f(t) f(it)$.

$f(t)$ is known by (6), therefore $g(z)$ is given by (7) and we obtain the solution of (8).

Note that the domains of integration in (5), (6) and (7) are $(-1, 1)$ for t and $(0, \infty)$ for z.

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