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A NOTE ON Γ-SEMIFIELD

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Abstract: The purpose of this short note is to obtain some necessary and sufficient conditions for a Γ-semiring to be a Γ-semifield.

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1. Introduction

In 1964, the notion of $\Gamma-$ring was introduced by N. Nobusawa [4] to provide algebraic home to Hom(A,B) and Hom(B,A) where A, B are additive Abelian groups. The notion of semiring was introduced by H.S. Vandiver [7] in 1934. $\Gamma-$semiring was introduced by M.M.K. Rao [5] in 1995 as a generalization of semiring as well as of $\Gamma-$ring and subsequently it was studied by Dutta and the present author [1], [2], [3]. It is well known that non-negetive cone R_0^+ of a totally ordered ring R forms a semiring with respect to the binary operations induced by those of R. It is surprising to note that in the non-positive cone R_0^- of R the induced multiplication is no longer closed and consequently it does not form a semiring with respect to the induced operations. But R_0^- becomes a Γ-semiring S where $S = \Gamma = R_0^-$. In fact R_0^- forms a Γ-semifield.

The notion of Γ-semifield was introduced by Dutta and Sardar in [2]. They obtained some characterizations of Γ-semifield via operator semirings of a Γ-semiring. The purpose of this short note is to obtain some more characterizations of Γ-semifields. Among other results we deduce that a commutative Γ-semiring S is a Γ-semifield if and only if S_α is a semifield for all $\alpha \in \Gamma - \{0\}$.

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2. Preliminaries

We recall the following preliminaries to develop the paper.

Definition 2.1. (see [5]) Let S and Γ be additive commutative monoids. If there exits a mapping S×Γ×S → S (with (m,γ, n)→ mγn ∈ S), satisfying the following conditions:

For all m, n, p ∈ S and for all γ, μ ∈ Γ,
(*i*) $m\gamma(n+p) = m\gamma n + m\gamma p$; $(m+n)\gamma p = m\gamma p + n\gamma p$; $m(\gamma+\mu)n = m\gamma n + m\mu n$;
(*ii*) $m\gamma(n\mu p) = (m\gamma n)\mu p$;
(*iii*) $m\gamma 0 = 0\gamma m = 0$, 0 is the zero element of S and $m\theta n = 0$, θ is the zero element of Γ then S is called a Γ-semiring.

Example 1. Let S be the set of all non-negative integers and Γ be the set of all non-negative even integers. Then S is a Γ−semiring with respect to usual addition and multiplication of integers.

Example 2. Let S and Γ be respectively the collections of all functions from Q_0^+ to Z_0^+ and from Z_0^+ to Q_0^+, where $(Z_0^+, +)$ and $(Q_0^+, +)$ be respectively the monoid of nonnegative integers and monoid of nonnegative rational numbers. Then $(S, +)$ and $(\Gamma, +)$ are monoids, where '+' is the addition of mappings. We define for $f, g \in$S and $\alpha \in \Gamma$, $f\alpha g = f \circ \alpha \circ g$, '∘' is the usual function composition. Then S is a Γ−semiring.

If A and B are subsets of a Γ-semiring S and $\Delta \subseteq \Gamma$, we denote by $A\Delta B$, the subset of S consisting of all finite sums of the form $\sum_i a_i\alpha_i b_i$, where $a_i \in A, b_i \in B$ and $\alpha_i \in \Delta$.

Definition 2.2. Let S be a Γ-semigroup. An additive subsemigroup I of S is called a left(right) ideal of S if SΓI⊆I (IΓS⊆I). If I is both a left ideal and right ideal then I is called a two-sided ideal or simply an ideal of S.

Definition 2.3. A Γ−semiring is said to be commutative if $a\alpha b = b\alpha a$ for all $a, b \in$S, $\alpha \in \Gamma$.

Definition 2.4. A Γ−semiring is said to be zero divisor free (ZDF) if $a\alpha b = 0$ implies that either $a = 0$ or $\alpha = 0$ or $b = 0$ where $a, b \in$S and $\alpha \in \Gamma$.

Definition 2.5. A commutative Γ−semiring S is said to be Γ−semifield if for any $a(\neq 0) \in$S and for any $\alpha \in \Gamma$ there exists $b \in$S, $\beta \in \Gamma$ such that $a\alpha b\beta d = d$ for all $d \in$S.

Example 3. Let $S = \Gamma = Q_0^-$ where Q_0^- is the set of all non-positive rational numbers. Then S is a Γ−semifield with respect to usual addition and multiplication of reals.

Definition 2.6. (see [6]) A Γ-semigroup S is said to be simple if it has no proper ideals.

Theorem 2.7. (see [6]) *Let S ba Γ-semigroup. Then S_α is a group if and only if S is a simple Γ-semigroup.*

Corollary 2.8. (see [6]) *Let S ba Γ-semigroup. If S_α is a group for some $\alpha \in \Gamma$ then S_α is a group for all $\alpha \in \Gamma$.*

Definition 2.9. (see [6]) A Γ-semigroup S is called a Γ-group if S_α is a group for some (and hence for all) $\alpha \in \Gamma$.

Combining Definition 2.6, Theorem 2.7, Corollary 2.8 and Definition 2.9 we easily deduce the following theorem.

Theorem 2.10. *A Γ-semigroup S is Γ-group if and only if S has no proper ideals.*

Theorem 2.11. (see [2]) *A commutative $\Gamma-$semiring S is a $\Gamma-$semifield if and only if S is ZDF and does not possess any nontrivial proper ideal.*

Definition 2.12. (see [2]) Let S be a Γ-semiring and F be the free additive commutative semigroup generated by S×Γ. Then the relation ρ on F, defined by $\sum_{i=1}^{m}(x_i,\alpha_i)\rho\sum_{j=1}^{n}(y_j,\beta_j)$ if and only if $\sum_{i=1}^{m}x_i\alpha_i a=\sum_{j=1}^{n}y_j\beta_j a$ for all a ∈S (m, n∈ Z^+) is a congruence on F. Congruence class containing $\sum_{i=1}^{m}(x_i,\alpha_i)$ is denoted by $\sum_{i=1}^{m}[x_i,\alpha_i]$. Then F/$\rho$ is an additive commutative semigroup. Now F/ρ forms a semiring with the multiplication defined by

$$\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)\left(\sum_{j=1}^{n}[y_j,\beta_j]\right)=\sum_{ij}[x_i\alpha_i y_j,\beta_j].$$

This semiring is denoted by L and called the left operator semiring of the Γ-semiring S.

Dually the right operator semiring R of the Γ-semiring S has been defined where R=$\left\{\sum_{i=1}^{m}[\alpha_i,x_i]:\alpha_i\in\Gamma,x_i\in \mathrm{S},i=1,2,....m;m\in \mathrm{Z}^+\right\}$ and the multiplication on R is defined as

$$\left(\sum_{i=1}^{m}[\alpha_i,x_i]\right)\left(\sum_{j=1}^{n}[\beta_j,y_j]\right)=\sum_{ij}[\alpha_i,x_i\beta_j y_j].$$

Theorem 2.13. (see [2]) *A ZDF, commutative $\Gamma-$semiring S is a $\Gamma-$semifield if and only if the left operator semiring L (the right operator semiring R) is a semifield.*

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3. Main Results

Theorem 3.1. *Let S be a ZDF commutative Γ-semiring and L, R be respectively the left operator semiring and right operator semiring of S. Then L is a semifield if and only if R is a semifield.*

Proof. Suppose L is a semifield. Then by Theorem 2.13, S is a Γ-semifield. So by applying Theorem 2.13 again we see that R is a semifield. The reverse argument shows that if R is a semifield the L is a semifield. □

Remark 1. It is well known that if S is a commutative Γ-semiring then L and R are isomorphic as semirings via the mapping $\sum_{i=1}^{m}[x_i, \alpha_i] \to \sum_{i=1}^{m}[\alpha_i, x_i]$. Hence in the statement of the above theorem we may omit the condition ZDF.

Theorem 3.2. *A commutative ZDF Γ-semiring S is a Γ-semifield if and only if $\{0\}$ is a maximal ideal of S.*

Proof. Let S be a Γ-semifield. Then by Theorem 2.11, S has no proper non trivial ideals. Consequently, $\{0\}$ is a maximal ideal of S.

Conversely, suppose the commutative ZDF Γ-semiring S is such that $\{0\}$ is a maximal ideal of S. Then S does not possess any non trivial proper ideal. Consequently, by Theorem 2.11, S is a Γ-semifield. □

Proposition 3.3. *A Γ-semifield is ZDF.*

Proof. Let S be a Γ-semifield. Let for $a, b \in S, \alpha \in \Gamma, a\alpha b = 0$. Let $a \neq 0, \alpha \neq 0$. Then by hypothesis there exists $c \in S, \beta \in \Gamma$ such that $a\alpha c\beta d = d$ (cf. Definition 2.5) for all $d \in S$. So, in particular, $b = a\alpha c\beta b = a\alpha b\beta c$ (using commutativity of S) $0\beta c = 0$. By applying similar argument we deduce that $b \neq 0, \alpha \neq 0$ imply $c = 0$ and $a \neq 0, b \neq 0$ give $\alpha = 0$. This completes the proof. □

Theorem 3.4. *Let S be a commutative Γ-semiring. Then S is aΓ-semifield if and only if S is ZDF and $S' = S - \{0\}$ is a Γ-group.*

Proof. Let the commutative Γ-semiring S be a Γ-semifield. and $a, b \in S'$ and $\alpha \in \Gamma'$. Then by Proposition 3.3, S is ZDF. Hence $a\alpha b \neq 0$ whence $a\alpha b \in S'$. Hence S' is a Γ'-semigroup. Let P be an ideal of the Γ'-semigroup S' and $a \in P$. Then $a \neq 0$. So for $\alpha \in \Gamma'$ there exist $b \in S, \beta \in \Gamma$ such that $a\alpha b\beta s = s$ for all $s \in S$. Clearly, $b \in S'$ and $\beta \in \Gamma'$. Thus we see that $s = a\alpha b\beta s \in P \forall s \in S$. Hence $P = S'$. Consequently, S' is a Γ'-group.

Conversely, suppose S' is a Γ'-group. Let $P(\neq 0)$ be an ideal of the Γ-semiring S and let $a(\neq 0) \in P$. Then $a \in S'$. By hypothesis, for any $\alpha \in \Gamma', S'_\alpha$ is a group (cf. Definition 2.9). Hence there exists $a' \in S'_\alpha$ such that $a\alpha a'$ is the identity of S'_α. Hence we obtain $a\alpha a'\alpha s = e\alpha s = s$ for all $s \in S'_\alpha = S'$. Again $a\alpha a'\alpha 0 = 0$. Hence we deduce that $a\alpha a'\alpha s = s$ for all $s \in S$. Further P being ideal of $S, a\alpha a'\alpha s \in P$ for all $s \in S$. Consequently, $P = S$. Hence by Theorem 2.11, S is a Γ-semifield. □

Theorem 3.5. *A commutative* Γ*-semiring S is a* Γ*-semifield if and only if for any non-zero element a of S and for $\alpha \in \Gamma'$ there exists an element $a' \in S$ such that $a\alpha a'\alpha b = b$ for all $b \in S$.*

Proof. Let S be a Γ-semifield and let $a \in S' = S - \{0\}$ and $\alpha \in \Gamma'$. By the above theorem S' is a Γ'-group. Hence S'_α is a group. So there exists $a' \in S'_\alpha$ such that $a\alpha a' - e$, the identity of S'_α. Then $a\alpha a'\alpha b = e\alpha b = b$ for all $b \in S'_\alpha = S'$. Also $a\alpha a'\alpha 0 = 0$. Consequently, $a\alpha a'\alpha b = b$ for all $b \in S$.

Converse follows from the definition of Γ-semifield. □

It is well known that each $\alpha \in \Gamma$ in a Γ-semigroup S or a Γ-semiring S gives rise to a multiplication such that S becomes a semigroup or a semiring. To conclude the paper we obtain the following result as an easy consequence of the above theorem which characterizes a Γ-semifield S in terms of the semiring S_α for all $\alpha \in \Gamma' = \Gamma - \{0\}$.

Corollary 3.6. *A commutative* Γ*-semiring S is a* Γ*-semifield if and only if S_α is a semifield for all $\alpha \in \Gamma'$.*

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