International Electronic Journal of Pure and Applied Mathematics
Volume 1 No. 1 2010, 61-74

ANALYSIS OF NONLINEAR ORBITING EARTH-SATELLITE PITCH ATTITUDE LIBRATION EQUATIONS

Emmanuel Osei-Frimpong[1], Anthony Y. Aidoo[2 §]

[1]Department of Mathematics
Kwame Nkrumah University of Science and Technology
Kumasi, GHANA
e-mail: eofrimp@yahoo.com

[2]Department of Mathematics and Computer Science
Eastern Connecticut State University
83, Windham Street, Willimantic, CT 06226, USA
e-mail: aidooa@easternct.edu

Abstract: We analyze a highly nonlinear system of equations that govern the pitch attitude librations of an Earth-satellite. Local stability of the linearized system is established and the analytical solution of the linearized equation which is usually obtianed numerically is expressed in terms of periodic functions.

AMS Subject Classification: 70F05, 85-02
Key Words: analysis, linearization, Earth-satellite, dynamics, Floquet theory

1. Introduction

Earth-satellites are designed to perform one or more of several important functions. This includes the control of space missions, meteorological and environmental monitoring, as well as several other applications such as television transmission by satellite which depend critically on Earth-satellites. In addition to the functions already listed, Earth-satellites have also provided a means of obtaining data on the density of the upper atmosphere and the Earth's gravitational field [4]. This mandates that the analysis of systems governing the motions of Earth-satellites be handled with all the accuracy that they deserve.

Several forces affect the motion of Earth satellites [2]. Consequently, equations

Received: April 14, 2010

§Correspondence author

that govern their dynamics and similar physical systems tend to be highly nonlinear [2]-[6]. In lieu of obtaining analytical or even numerical solutions to such equations, various techniques such as linearization or phase plane analysis are usually employed to extract the needed information from the equations.

Dynamical systems theory provides techniques for the analysis of model equations and the classification of the resulting solutions. Floquet theory has been applied successfully in the analysis of several dynamical systems in which time periodic functions are involved [13]. The Floquet method provides a clear understanding of the motion of the physical system [8]. However, its application to system of equations governing Earth-satellite motions such as being studied in this paper has not been fully investigated. Thus, we approach the analysis of this highly nonlinear system in a novel method in dynamical systems theory. This enables us to express the solution in terms of periodic functions, a more accurate derivation than what is usually available numerically.

2. Governing Equations

Following Osei-Frimpong et al [11], we otain the equation of motion of the satellite in terms of its anomaly as follows. We express the attitude librations of an Earth-satellite in terms of the roll, pitch and yaw (Euler angles) [5], [6]. The equations are then derived by making the following assumptions:

1. The Earth is spherically symmetrical and is of uniform density and thus it is treated as a point mass.

2. The satellite orbits high enough above the Earth, as such, drag force is negligible.

3. There is no maneuvering or significant change in path, hence thrust force is ignored.

4. Other forces such as those due to solar radiation and electromagnetic fields are negligible compared to the Earth's gravity.

5. Since the satellite is relatively close to the Earth, the gravitational attraction of the sun and other third bodies are ignored.

The list symbols and parameters used in the model and analysis are given below.

Symbol	Description
r	Orbital radius
μ	Gravitational parameters
ϕ	Roll attitude angle
ψ	Yaw attitude angle
θ	Pitch attitude angle
ω_c	Orbital rate corresponding to circular orbit at perigee
e	Eccentricity of the orbit

Symbol	Description
φ	True anomaly of the satellite
σ	Inertial ratio
F_g	Gravitational force on satellite due to Earth
G	Universal gravitational constant
m_e	Mass of Earth
m	Mass of satellite
r	Distance from satellite to Earth's center
$\hat{r}$	Unit vector in the direction or the satellite position from the center of Earth

Ignoring the departure of motion from Newton's law as a result of relativity effects [3], we derive the equation that describes the motion of the satellite by beginning with Newton's Universal law of gravitation. This leads to the equation:

$$F_g = -\frac{Gm_e m}{r^2}\hat{r}. \tag{1}$$

Using Newton's second law of motion, we substitute for F_g in equation (1) to obtain:

$$m\ddot{r} = -\frac{\mu m}{r^3} r, \tag{2}$$

where $\mu = Gm_e$ is a gravitational parameter. Thus the orbital equation of motion is given by

$$\ddot{r} + \frac{\mu}{r^3} r = 0. \tag{3}$$

Before we proceed, another critical asumption is required here. The satellite is assumed to be a rigid non-spinning body with no internal momentum and no passive or active control subsystem, while the Earth is considered as a point mass. Hence, the Earth and orbiting satellite become a two body dynamical system. Based on these assumptions, the attitude dynamics can be described by the dynamical equations describing a rigid body. Thus, we apply Newtonian universal law of gravitation and ignore the departure of the motion from Newton's law due to relativity effects [4]. This leads to the following governing equations:

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= M_x, \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= M_y, \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= M_z, \end{aligned} \tag{4}$$

where I_x, I_y, I_z are the non-zero components of the moment of inertia tensor of the body, and M_x, M_y, M_z are components of the external torque vector acting on the body. The external torque attributed to gravity is usually expressed as:

$$M_x = -\frac{3\mu}{r_c^3}(I_y - I_z)\cos^2\theta\cos\phi\sin\phi,$$

$$\begin{aligned} M_y &= -\frac{3\mu}{r_c^3}(I_x - I_z)\sin\theta\cos\phi\cos\phi, \\ M_z &= -\frac{3\mu}{r_c^3}(I_y - I_x)\sin\theta\cos\phi\sin\phi. \end{aligned} \tag{5}$$

In order to obtain the attitude equations we substitute (2) in (1) and replace the angular velocities and their derivatives to obtain:

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= -\frac{3\mu}{r_c^3}(I_y - I_z)\cos^2\theta\cos\phi\sin\phi, \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_x\omega_z &= -\frac{3\mu}{r_c^3}(I_x - I_z)\sin\theta\cos\theta\cos\phi, \\ I_z + (I_y - I_x)\omega_x\omega_y &= -\frac{3\mu}{r_c^3}(I_y - I_x)\sin\theta\cos\theta\sin\phi. \end{aligned} \tag{6}$$

Satellite motions (librations) are known to consist of a combination of relatively minute rotational motions compared to the main orbital motion [10]. Consequently, the dynamics of the physical system controlling the motion are left intact when terms representing such small motions are neglected. This enables us to neglect reatively inessential terms of degree greater than or equal to two which are of lower order in terms of accuracy. Therefore the dynamical equations become:

$$\begin{aligned} I_x\ddot{\phi} - I_x\dot{\omega}_0\sin\Psi - \omega_0(I_z - I_y + I_x) - (I_z - I_y)(\omega_0^2 + \frac{3\mu}{r_c^3})\sin\phi &= 0, \\ I_y\ddot{\theta} + \frac{3\mu}{r_c^3}(I_x - I_z)\sin\theta &= I_y\dot{\omega}_0, \\ I_z\dot{\psi} + I_z\dot{\omega}_0 + \omega(I_z - I_y + I_x)\dot{\phi} - \omega_0^2(I_x - I_y)\sin\psi &= 0, \end{aligned} \tag{7}$$

where ϕ is the roll attitude angle, ψ is the yaw attitude angle, ω_c is the orbital rate corresponding to a circular orbit at the perigee and θ is the pitch attitude angle.

From the system of equations (7) we note that the dynamics comprise of a uncoupled pitch mode and a coupled roll-yaw mode. This makes it possible to study the two modes independently. Our approach considers the pitch attitude. The model equation then consists of the orbital equation (3) and the pitch attitude equation from the system (7). These equations can be transformed into state-space form by making the following substitutions:

$$x_1 = \theta, \tag{8}$$

$$\dot{x_1} = \dot{\theta} = x_2, \tag{9}$$

$$\alpha_1 = \frac{\omega_c^2}{2}\Big(\frac{1 + e\cos\varphi}{1 + e}\Big)^3, \tag{10}$$

$$\alpha_2 = e\sin\varphi, \tag{11}$$

where φ is the true anomaly [9].

We obtain the final form of our model equations by eliminating time t from the system of first order equations that results from the substitutions above. The resulting is in terms of the true anolomaly φ [7], [12]. By the chain rule of differentiation we have:

$$\dot{x}_i = \frac{dx_i}{dt} = \frac{dx_i}{d\varphi}\frac{d\varphi}{dt} = \dot{\varphi}x_i', \quad i = 1, 2. \tag{12}$$

Since

$$\dot{\varphi} = \frac{\omega_c^2(1 + e\cos\varphi\varphi)^2}{(1+e)^{3/2}}, \tag{13}$$

we have:

$$x_1' = \frac{\dot{x}_1}{\dot{\varphi}} = \frac{(1+e)^{3/2}}{\omega_c(1+e\cos\varphi)^2}x_2,$$

$$\begin{aligned} x_2'(\varphi) &= \frac{(1+e)^{3/2}}{\omega_c(1+e\cos\varphi)^2}\Big[-3\alpha_1\sigma\sin 2x_1 - 4\alpha_1\alpha_2\Big] \\ &= \frac{(1+e)^{3/2}}{\omega_c(1+e\cos\varphi)^2}3\alpha_1\omega\Big[-\sin 2x_1 - \frac{4\alpha_2}{3\sigma}\Big] \\ &= \frac{3\sigma\omega_c(1+e\cos\varphi)}{2(1+e\cos\varphi)^{3/2}}\Big[-\sin 2x_1 - \frac{4\alpha_2}{3\sigma}\Big]. \end{aligned}$$

By making the substitution $k_1 = \frac{1}{k_2} = \frac{(1+e)^{3/2}}{\omega_c}$ the system of nonlinear equations reduces to

$$x_1'(\varphi) = \frac{k_1}{(1+e\cos\varphi)^2}x_2, \tag{14}$$

$$x_2'(\varphi) = \frac{3\sigma k_2}{2}(1+\cos\varphi)(-2\sin 2x_1 - \frac{4e}{3\sigma}\sin\varphi). \tag{15}$$

The two equations (14, 15) making up the system can be expressed as a single vector equation as:

$$x'(\varphi) = F(x, \varphi), \tag{16}$$

where

$$x(\varphi) = \begin{bmatrix} x_1(\varphi) \\ x_2(\varphi) \end{bmatrix},$$

and

$$F(x,\varphi) = \begin{bmatrix} \frac{k_1}{(1+e\cos\varphi)^2}x_2 \\ \frac{3\sigma k_2}{2}(1+e\cos\varphi)(-2\sin 2x_1 - \frac{4e}{3\sigma}\sin\varphi) \end{bmatrix}.$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

3. Analysis

3.1. Equilibrium States

The analysis of the system equations representing the attitude dynamics of the satellite involves the use of dynamical systems theory. In most other applications, such equations have constant coefficients. In our case, the linearized equations we obtain have periodic coefficients. This leads to the use of Floquet theory version of linear systems theory.

We begin our analysis with finding the steady states. For the system being considered here, the equilibrium states are obtained when the right hand side of the state space equations vanish. This means that:

$$x_2 = 0. \tag{17}$$

Hence:

$$(1 + e\cos\varphi)(-\sin 2x_1 - \frac{4e}{3\sigma}\sin\varphi) = 0.$$

This implies that: $1 + e\cos\varphi = 0$ or $\sin 2x_1 = -\frac{4e}{3\sigma}\sin\varphi) = -\frac{4\alpha_2}{3\sigma}$. That is

$$1 + e\cos\varphi = 0 \text{ or } x_1 = \frac{1}{2}\sin^{-1}\big(-\frac{4\alpha_2}{3\sigma}\big).$$

For closed eccentric orbits we have:

$$0 \le e < 1 \text{ and also } 0 \le |\cos\varphi| \le 1.$$

Hence $1 + e\cos\varphi \neq 0$. The equilibrium states are therefore given by

$$(\bar{x_1}, \bar{x_2}) = \Big(\frac{1}{2}\sin^{-1}\big(-\frac{4\alpha_2}{3\sigma}\big), 0\Big). \tag{18}$$

3.2. Linearization

Linearization enables us to investigate the local stability of the equilibrium points. We can only give a global description of the nonlinear equations using the nullclines. Although the validity of the linearized model applies only when the system operates the within a small range of the steady states, linearization makes it possible to deduce more information about the nonlinear system that would otherwise had not been possible. We express the state space equations in the form:

$$\begin{aligned} x_1' &= f(x_1, x_2), \\ x_2' &= g(x_1, x_2), \end{aligned} \tag{19}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

where

$$f(x_1, x_2) = \frac{k_1}{(1 + e\cos\varphi)^2} x_2, \tag{20}$$

$$g(x_1, x_2) = \frac{3\sigma k_2}{2}(1 + e\cos\varphi)(-2\sin 2x_1 - \frac{4e}{3\sigma}\sin\varphi). \tag{21}$$

At the steady state, the Jacobian matrix for the system above is given by:

$$J(\bar{x_1}, \bar{x_2}) = \begin{bmatrix} 0 & \frac{k_1}{(1+e\cos\varphi)^2} \\ -3\sigma k_2(1 + e\cos\varphi)(\cos 2\bar{x_1}) & 0 \end{bmatrix}.$$

The linearized equations are given by:

$$\Delta x_1' = 0 \cdot \Delta x_1 + \frac{1}{\omega_c(1 + e\cos\varphi)^2}\Delta x_2, \tag{22}$$

$$\Delta x_2' = -3\sigma k_2(1 + e\cos\varphi)(\cos 2\bar{x_1}) \cdot \Delta x_1 + 0 \cdot \Delta x_2. \tag{23}$$

At the equilibrium point the eigenvalues λ of the linearized matrix are given by

$$\lambda^2 = -\frac{3k_1k_2\sigma}{1 + e\cos\varphi}\cos\left(\sin^{-1}(\frac{-4\sin\varphi}{3\sigma})\right) < 0.$$

Hence λ's are pure complex conjugates. This confirms the fact that the equilibrium point is a center.

The linearized system represented by equations (22) and (23) may be expressed in matrix form as:

$$y'(\varphi) = \begin{bmatrix} 0 & \frac{k_1}{(1+e\cos\varphi)^2} \\ -3\sigma k_2(1 + e\cos\varphi)\cos[\sin^{-1}\left(\frac{-4e\sin\varphi}{3\sigma}\right)] & 0 \end{bmatrix} y(\varphi),$$

where

$$y(\varphi) = x - \bar{x} = \Delta x.$$

In standard notation, the linearized equations may be written as:

$$y'(\varphi) = A(\varphi)y(\varphi).$$

Here, the coefficient matrix $A(\varphi)$ is given by:

$$A(\varphi) = \begin{bmatrix} 0 & \frac{k_1}{(1+e\cos\varphi)^2} \\ -3\sigma k_2(1 + e\cos\varphi)\cos[\sin^{-1}\left(\frac{-4e\sin\varphi}{3\sigma}\right)] & 0 \end{bmatrix}.$$

We note that the matrix $A(\varphi)$ is periodic in φ with period 2π. Thus $A(\varphi+2\pi) = A(\varphi)$. For $A(\varphi)$ to be periodic in time t we must have: $A(\varphi(t + T)) = A(\varphi(t))$. It suffices to show that that there exists a real number T such that:

$$\varphi(t + T) = \varphi(t) + 2\pi. \tag{24}$$

Using

$$\dot{\varphi} = \frac{\omega_c \sigma (1 + e\cos\varphi)^2}{(1+e)^{3/2}}, \tag{25}$$

we have

$$\frac{d\varphi}{(1+e\cos\varphi)^2} = \frac{\omega_c\sigma}{(1+e)^{3/2}}dt. \tag{26}$$

Hence

$$\int \frac{d\varphi}{(1+e\cos\varphi)^2} = k\int dt, \tag{27}$$

where

$$k = \frac{\omega_c\sigma}{2(1+e)^{3/2}}.$$

Evaluating the integrals in (27) leads to the following equation:

$$\frac{e\sin\varphi}{(e^2-1)(1+e\cos\varphi)} - \frac{2\alpha(\varphi)}{(e^2-1)(1-e^2)^{1/2}} = kt + c, \tag{28}$$

where c is the constant of integration and

$$\alpha(\varphi) = \tan^{-1}\left[\left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{1}{2}\varphi\right].$$

Let $\varphi(t_0) = -\pi$ and $\varphi(t_0 + T) = \pi$. Then evaluating equation (28) at t_0 and $t_0 + T$ and subtracting one from the other, we get:

$$kT = \frac{2\{\alpha(-\pi) - \alpha(\pi)\}}{(e^2-1)(1-e)^{1/2}}. \tag{29}$$

Hence

$$T = \frac{2\{\alpha(-\pi) - \alpha(\pi)\}}{k(e^2-1)(1-e)^{1/2}}. \tag{30}$$

This shows that the coefficient matrix of the linearized system of equations is periodic with respect to time t and has period T is given by equation (30).

3.3. Floquet Theory Analysis

The periodicity of the coefficients of the linearized system enables us to use Floquet theory for further analysis of the system [8], [13]. Writing the linearized system in the form:

$$x' = A(\varphi)x, \tag{31}$$

where $A(\varphi)$ is an $n \times n$ piecewise continuous matrix valued function such that $A(\varphi + 2\pi) = A(\varphi)$. By Floquet theory, the solution of the linearized system is given by:

$$x = Q(\varphi)e^{R\varphi}x_0, \tag{32}$$

with $Q(0) = I$, and Q and R are $n \times n$ matrices. Differentiating equation (32) with respect to φ and substituting in equation (31), we obtain:

$$(Q'(\varphi) + Q(\varphi)R)e^{R\varphi\varphi}x_0 = A(\varphi)Q(\varphi)e^{R\varphi}x_0.$$

This reduces to:

$$Q'(\varphi) + Q(\varphi)R - A(\varphi)Q(\varphi) = 0. \quad (33)$$

Since $A(\varphi)$ and $Q(\varphi)$ are both continuous and periodic of period 2π, we can expand each in Fourier series. Thus

$$\begin{aligned} A(\varphi) &= a_0 + \sum_{j=1}^{\infty}[a_j \cos(j\varphi) + b_j \sin(j\varphi)], \quad (34) \\ Q(\varphi) &= p_0 + \sum_{j=1}^{\infty}[p_j \cos(j\varphi) + q_j \sin(j\varphi)], \quad (35) \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\varphi)d\varphi, \\ a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} A(\varphi)\cos(j\varphi)d\varphi, \quad (36) \\ b_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} A(\varphi)\sin(j\varphi)d\varphi. \end{aligned}$$

The p_j's and q_j's are defined in a similar manner. Substituting equations (34) and (35) in equation (33), we obtain:

$$\begin{aligned} & \sum_{j=1}^{\infty} j[-p_j \sin(j\varphi) + q_j \cos(j\varphi)] + p_0 R + \sum_{j=1}^{\infty} j[p_j \cos(j\varphi) + q_j \sin(j\varphi)]R \\ - \; & a_0 p_0 - a_0 \sum_{j=1}^{\infty} j[p_j \cos(j\varphi) + q_j \sin(j\varphi)] - p_0 \sum_{j=1}^{\infty} j[a_j \cos(j\varphi) + b_j \sin(j\varphi)] \\ - \; & \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}[a_r \cos(r\varphi) + b_r \sin(\varphi r\varphi)][p_s \cos(s\varphi) + q_s \sin(s\varphi)] = 0. \end{aligned}$$

This can be reduced to:

$$\begin{aligned} & p_0 R - a_0 p_0 - \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_r + b_r q_r) \\ + \; & [(p_1 R + q_1 - a_0 p_1 - a_1 p_1) - \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_{r+1} + b_r q_{r+1}) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_{r-1} + b_r q_{r-1})]\cos\varphi + [q_1 R - p_1 - a_0 q_1 - b_1 p_0) \\
&- \quad \frac{1}{2}\sum_{r=1}^{\infty}(a_r q_{r+1} + b_r p_{r+1})] + \sum_{j=2}^{\infty}[\{(p_j R + j q_j - a_0 p_j - a_j p_0) \\
&+ \quad \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_{j-r} - b_r q_{j-r})] - \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_{r+j} + b_r q_{r+j}) \\
&+ \quad \frac{1}{2}\sum_{r=j+1}^{\infty}(a_r p_{j-r} + b_r q_{j-r})\}\cos j\varphi \\
&+ \quad \{(q_j R - j p_j - a_0 q_j - b_j p_0) - \frac{1}{2}\sum_{r=1}^{\infty}(a_r q_{j-r} + b_r p_{j-r}) \\
&- \quad \frac{1}{2}\sum_{r=1}^{\infty}(a_r q_{r+j} + b_r p_{r+j}) - \frac{1}{2}\sum_{r=j+1}^{\infty}(-a_r q_{r-j} + b_r p_{r-j})\}\sin j\varphi = 0.
\end{aligned}
$$

In order to obtain a set of indicial and recurrence relations that will enable us to determine the coefficients of the Fourier series expansion of the Floquet periodic matrix Q, we equate the coefficients of the circular functions to zero. The resulting five equations are:

(i) $p_0 R - a_0 p_0 - \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_r + b_r q_r) = 0.$

(ii) $(p_1 R + q_1 - a_0 p_1 - a_1 p_0) - \frac{1}{2}\sum_{r=1}^{\infty}(a_r p_{r+1} + b_r q_{r+1})$
$$+\frac{1}{2}\sum_{r=1}^{\infty}(a_r p_{r-1} + b_r q_{r-1}) = 0.$$

(iii) $(q_1 R - p_1 - a_0 q_1 - b_1 p_0) - \frac{1}{2}\sum_{r=1}^{\infty}(a_r q_{r+1} + b_r p_{r+1})$
$$+\frac{1}{2}\sum_{r=2}^{\infty}(-a_r q_{r-1} + b_r p_{r-1}) = 0.$$

(iv) $(p_j R + j q_j - a_0 p_j - a_j p_0) + \frac{1}{2}\sum_{r=1}^{\infty}[a_r(p_{j-r} p_{j+r}) - b_r(q_{j-r} + q_{j+r})]$
$$-\frac{1}{2}\sum_{r=j+1}^{\infty}(a_r p_{r-j} + b_r q_{r-j}) = 0 \text{ for } j = 2, ..., \infty.$$

(v) $(q_j R - j p_j - a_0 q_j - b_j p_0) - \frac{1}{2}\sum_{r=1}^{\infty}[a_r(q_{j-r} - q_{j+r}) - b_r(p_{j-r} - p_{j+r})]$

$$-\frac{1}{2}\sum_{r=j+1}^{\infty}(-a_r q_{r-j} + b_r p_{r-j}) = 0 \text{ for } j = 2, ..., \infty.$$

The infinite set of nonlinear equations in p_i, q_i and R that appear in (i) through (v) provide adequate information to enable us to determine the coefficients of the Fourier series expansion of the Floquet periodic matrix up to any number of terms of truncation. This information is also adequate enough to determine the coefficients of the corresponding Floquet constant matrix R.

4. Results

We have established that the solution to the linearized equations obtained from the nonlinear system describing the Earth-satellite pitch librations can be expressed in the form of Fourier series. This is given in the form:

$$x(\varphi) = \sum_{j=0}^{\infty}[\mu_j(\varphi)\cos j\varphi + \eta_j(\varphi)\sin j\varphi],$$

where

$$\mu_j(\varphi) = p_j e^{R\varphi} x_0, \qquad \eta_j(\varphi) = q_j e^{R\varphi} x_0.$$

The analysis of the linear system does not only determine the stability or instability of the invariant set but also determines the phase portrait of the linear system. The resulting phase portrait is an approximation to the phase portrait to the nonlinear system and is valid in the neighborhood of the invariant set. The solution to the linearized system for our model is displayed as a phase-plane diagram in Figure 1.

The corresponding anomaly history plot is shown in Figure 2.

5. Conclusion

We have analyzed the local dynamics of a system of nonlinear equations which represent an Earth-satellite pitch attitude librations. The equation of motion of the pitch attitude is first transformed in terms of the true anomaly. Using dynamical systems theory, the system of equations is analyzed for its equilibrium and then periodicity by linearizing about a fixed point. It is shown that the linearized system possess periodic coefficients. Appealing to Floquet theory, the nonlinear solution is expressed in terms of series, of which the coefficients are easily obtained.

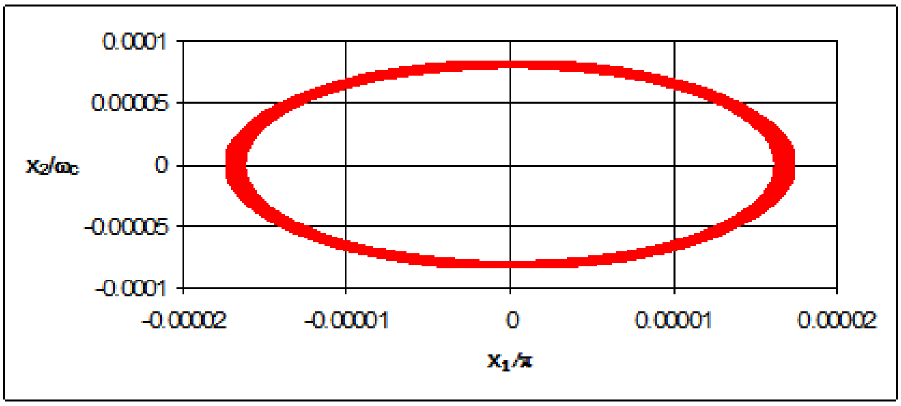

Figure 1: Phase-plane diagram of the linear solution

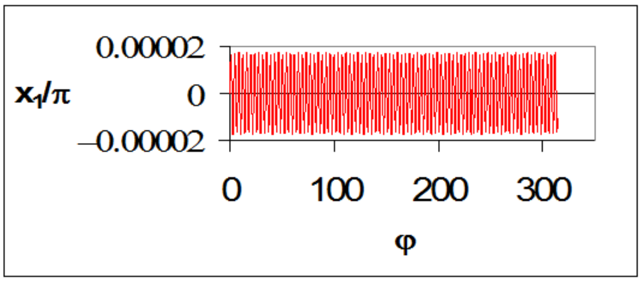

Figure 2: Anomaly history of the linear solution

Acknowledgments

This paper is dedicated to the memory of the late Professor Anthony K. Amos, who introduced the first author to the topic. The first author is also indebted to the Department of Aerospace Engineering, Pennsylvania State University, University Park, PA, USA, for welcoming him to undertake his PhD dissertation. The core of this paper is based on the dissertation.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

References

[1] A.K Amos, On spacecraft tumbling, *Journal of Chinise Society of Mechanical Engineering*, **23** (2000), 41-47.

[2] K.B. Bhatnagar, M. Mehra, The motion of a geosynchronous satellite - I, *Indian Journal of Pure and Applied Mathematics*, **17** (1986), 1438-1452.

[3] V.A. Brumberg, On relativistic equations of motion of an Earth satellite, *Celestial Mechanics and Dynamical Astronomy*, **88** (2004), 209-225.

[4] G.V. Groves, Motion of a satellite in the Earth's gravitational field, *Proceedings of the Royal Society of London Series A, Mathematical Physical Sciences*, **254** (1960), (1960).

[5] M. Inarrea, V. Lanchares, Chaotic pitch motion of an asymmetric non-rigid spacecraft with viscous drag in circular orbit, *International Journal of Nonlinear Mechanics*, **36** (2006), 86-100.

[6] K.D. Kumar, Satellite attitude stabilization using fluid rings, *Acta Mechanica*, **208** (2009), 117-131.

[7] K.D. Kumar, M.J. Tahk, H.C. Bang, Satellite attitude stabilization radiation pressure using solar radiation and magnetotorquer, *Control Engineering Practice*, **17** (2009), 267-279.

[8] C. Lee, D. Han, S. Hong, Modal analysis of periodically time-varying linear rotor systems using Floquet theory, In: *7-th IFToMM-Conference on Rotor Dynamics*, Vienna, Austria (2006), 1-9.

[9] A.K. Misra, Dynamics and control of tethered satellite systems, *Acta Astronautica*, **63** (2008), 1169-1177.

[10] A.M. Si Mohammed, M. Benyettou, Y. Bentoutou, A. Boudjemai, Y. Hashida, M.N. Sweeting, Three-axis active control system for gravity gradient stabilized microsatellite, *Acta Astronautica* **64** (2009), 796-809.

[11] E. Osei-Frimpong, A.Y. Aidoo, A.K. Amos, Analytical solution to a highly nonlinear Earth-satellite pitch attitude libration equation, *International Journal of Pure and Applied Mathematics*, **58** (2010), 425-434.

[12] T.R. Patel, K.D. Kumar, K. Behdinan, Variable structure control for satellite attitude stabilization in elliptic orbits using solar radiation pressure, *Acta Astronautica*, **64** (2009), 359-773.

[13] D. Viennot, Geometric phases in adiabatic Floquet theory, Abelian gerbes and Cheon's anholonomy, *Journal of Physics A: Mathatical and Theoretical*, **42** (2009), 1-22.

[14] B. Wong, A. Misra, Planar dynamics of variable length multi-tethered spacecraft near collinear Lagrangian points, *Acta Astronautica*, **63** (2008), 1178-1187.