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EXAMINATION OF MIXED PROBLEM WITH PERIODIC BOUNDARY CONDITION FOR A CLASS OF QUARTIC PARTIAL DIFFERENTIAL QUASI-LINEAR EQUATION

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Abstract: A mixed problem with periodic boundary condition $\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = f(t, x, u)$ is examined for quasi-linear quartic partial differential equation. The existence, uniqueness and continuity of weak generalized solution is proved. Firstly, test function is defined for solution of problem, then the generalized (weak) solution of the problem is defined by this function. Then weak solution is searched as a Fourier series with unknown variable coefficients, then for these coefficients a system of infinite integral equations is obtained. The existence and uniqueness of this system is proved by consecutive approximation method in B_T-Banach space. Finally the norm of the difference between exact and approximate solutions of the system in B_T is obtained.

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1. Introduction

Examination of quartic, linear and quasi-linear equations is important for vibration analysis of bars in terms of engineering. In this study, the existence and uniqueness of the generalized (weak) solution of a mixed problem with quasi-linear, quartic equation is examined by nonlinear Fourier method. The existence of solutions which

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are done in this study show that using of this approach in engineering mathematics especially dealing with nonlinear vibrations of bars, dynamic stability of bars formed by composite material and carbon nanotube is more effective.

2. Formulation of Problem

We consider the following mixed problem

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^4 u}{\partial x^4} = f(t,x,u), \quad (t,x) \in D\{0 < t < T,\ 0 < x < \pi\}, \tag{2.1}$$

$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \quad (0 \le x \le \pi), \tag{2.2}$$

$$\begin{gathered} u(t,0) = u(t,\pi),\ u_x(t,0) = u_x(t,\pi), \\ u_{x^2}(t,0) = u_{x^2}(t,\pi),\ u_{x^3}(t,0) = u_{x^3}(t,\pi) \quad (0 \le t \le T), \end{gathered} \tag{2.3}$$

where $\varphi(x), \psi(x)$ and $f(t,x,u)$ are given functions defined on $[0,\pi]$ and $\bar{D}\{0 \le t \le T,\ 0 \le x \le \pi\} \times (-\infty,\ \infty)$ respectively and the function $u(t,x)$ is solution of the problem.

Definition 2.1. The function $v(t,x) \in C(\bar{D})$ is called *test function* if it has continuous partial derivatives of order contained in equation (2.1) and satisfies both following conditions

$$v(T,x) = v_t(T,x) = 0$$

and the boundary condition (2.3).

From [1], we give the definition;

Definition 2.2. The function $u(t,x) \in C(\bar{D})$ satisfying the integral identity

$$\int_0^T \int_0^\pi \left\{ u \left[\frac{\partial^2 v}{\partial t^2} + a^2 \frac{\partial^4 v}{\partial x^4} \right] - f(t,x,u)v \right\} dx\, dt$$
$$\int_0^\pi \varphi(x) v_t(0,x)\, dx - \int_0^\pi \psi(x) v(0,x)\, dx = 0, \tag{2.4}$$

for an arbitrary test function $v(t,x)$ is called *weak generalized solution* of problem (2.1)-(2.3).

The set

$$\{\bar{u}(t)\} = \{\frac{1}{2} u_0(t), u_{c1}(t), u_{s1}(t), \dots, u_{ck}(t), u_{sk}(t), \dots\}$$

of continuous on $[0,T]$ functions satisfying the condition

$$\frac{1}{2} \max_{t\in[0,T]} |u_0(t)| + \sum_{k=1}^{\infty} \left[\max_{t\in[0,T]} |u_{ck}(t)| + \max_{t\in[0,T]} |u_{sk}(t)| \right] < \infty,$$

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denote by B_T. Let

$$\|\overline{u}(t)\|_{B_T} = \frac{1}{2}\max_{t\in[0,T]}|u_0(t)| + \sum_{k=1}^{\infty}\left[\max_{t\in[0,T]}|u_{ck}(t)| + \max_{t\in[0,T]}|u_{sk}(t)|\right]$$

be the norm in B_T. It can be shown that B_T is Banach space.

3. Solution of Problem

We look for a weak solution of problem (2.1)-(2.3) in the form

$$u(t,x) = \frac{1}{2}u_0(t) + \sum_{k=1}^{\infty}\left[u_{ck}(t)\cos 2kx + u_{sk}(t)\sin 2kx\right], \tag{3.1}$$

for the following unknown functions $u_0(t), u_{ck}(t), u_{sk}(t)$, $(k = \overline{1,\infty})$. In order to determinate unknowns using equation (2.4), we get the infinite system of integral equations;

$$u_0(t) = \varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau) f\left\{\tau,\xi,\frac{1}{2}u_0(\tau) + \sum_{n=1}^{\infty}\left[u_{cn}(\tau)\cos 2n\xi + u_{sn}(\tau)\sin 2n\xi\right]\right\}\, d\xi\, d\tau,$$

$$u_{ck}(t) = \varphi_{ck}\cos\alpha_k t + \frac{\psi_{ck}}{\alpha_k}\sin\alpha_k t + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\left\{\tau,\xi,\frac{1}{2}u_0(\tau) + \sum_{n=1}^{\infty}\left[u_{cn}(\tau)\cos 2n\xi + u_{sn}(\tau)\sin 2n\xi\right]\right\} \times \cos 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau,$$

$$u_{sk}(t) = \varphi_{sk}\cos\alpha_k t + \frac{\psi_{sk}}{\alpha_k}\sin\alpha_k t + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\left\{\tau,\xi,\frac{1}{2}u_0(\tau) + \sum_{n=1}^{\infty}\left[u_{cn}(\tau)\cos 2n\xi + u_{sn}(\tau)\sin 2n\xi\right]\right\} \times \sin 2k\xi \sin\alpha_k(t-\tau)\, d\xi\, d\tau,$$

$$\alpha_k = a(2k)^2,\ k = \overline{1,\infty}. \tag{3.2}$$

For the system (3.2), the following theorem is true.

Theorem 3.1. *Suppose the following conditions are satisfied:*
a) $f(t,x,u)$ *is continuous respect to all arguments on* $\overline{D}\times(-\infty,\infty)$.
b) $|f(t,x,u)-f(t,x,v)|\le b(t,x)|u-v|$ *where* $b(t,x)\in L_2(D),\ b(t,x)>0$.
c) $f(t,x,0)\in L_2(D)$.
d) The functions $\varphi(x),\ \psi(x)$ *with* $\varphi(x)\in C^1[0,\pi],\psi(x)\in C[0,\pi]$ *satisfy the following conditions:*

$$\varphi(0)=\varphi(\pi),\ \varphi'(0)=\varphi'(\pi),\ \psi(0)=\psi(\pi).$$

In this case, the system (3.2) has unique solution in B_T.

Proof. We use prove the method of successive approximation. For the system (3.2), let us define an iteration as follows

$$u_0^{(N+1)}(t)=u_0^{(0)}(t)$$
$$+\frac{2}{\pi}\int_0^t\int_0^\pi(t-\tau)f\{\tau,\xi,\frac{1}{2}\,u_0^{(N)}(\tau)+\sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(\tau)\cos 2n\xi+u_{sn}^{(N)}(\tau)\sin 2n\xi\Big]\}\ d\xi\, d\tau,$$

$$u_{ck}^{(N+1)}(t)=u_{ck}^{(0)}(t)$$
$$+\frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\{\tau,\xi,\frac{1}{2}\,u_0^{(N)}(\tau)+\sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(\tau)\cos 2n\xi+u_{sn}^{(N)}(\tau)\sin 2n\xi\Big]\}$$
$$\times\cos 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau, \quad (3.3)$$

$$u_{sk}^{(N+1)}(t)=u_{sk}^{(0)}(t)$$
$$+\frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f\{\tau,\xi,\frac{1}{2}\,u_0^{(N)}(\tau)+\sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(\tau)\cos 2n\xi+u_{sn}^{(N)}(\tau)\sin 2n\xi\Big]\}$$
$$\times\sin 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau,$$
$$N=\overline{0,\infty},$$

where $u_0^{(0)}(t)=\varphi_0+\psi_0 t,\ u_{ck}^{(0)}(t)=\varphi_{ck}\cos\alpha_k t+\frac{\psi_{ck}}{\alpha_k}\sin\alpha_k t, u_{sk}^{(0)}(t)=\varphi_{sk}\cos\alpha_k t+\frac{\psi_{sk}}{\alpha_k}\sin\alpha_k t,\ (k=\overline{1,\infty})$. For simplicity, letting

$$Au^{(N)}(t,\xi)=\frac{1}{2}\,u_0^{(N)}(t)+\sum_{n=1}^{\infty}\Big[u_{cn}^{(N)}(t)\cos 2n\xi+u_{sn}^{(N)}(t)\sin 2n\xi\Big]$$

and

$$\{\overline{u}^{(N)}(t)\}=\Big\{\frac{1}{2}\,u_0^{(N)}(t),u_{c1}^{(N)}(t),u_{s1}^{(N)}(t),\ldots,u_{cn}^{(N)}(t),u_{sn}^{(N)}(t),\ldots\Big\}$$

the successive approximations (3.2) turns out to be

$$\begin{aligned}
u_0^{(N+1)}(t) &= u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau) f[\tau,\xi,Au^{(N)}(\tau,\xi)]\ \ d\xi\, d\tau,\\
u_{ck}^{(N+1)}(t) &= u_{ck}^{(0)}(t) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(N)}(\tau,\xi)]\cos 2k\xi \sin\alpha_k(t-\tau)\ \ d\xi\, d\tau,\\
u_{sk}^{(N+1)}(t) &= u_{sk}^{(0)}(t) + \frac{2}{\pi\alpha_k}\int_0^t\int_0^\pi f[\tau,\xi,Au^{(N)}(\tau,\xi)]\sin 2k\xi \sin\alpha_k(t-\tau)\ \ d\xi\, d\tau\\
(k &= \overline{1,\infty}).
\end{aligned} \tag{3.4}$$

It is clear that

$$\max_{0\le t\le T}|Au^{(N)}(t,\xi)| \le \frac{1}{2}\max_{0\le t\le T}|u_0^{(N)}(t)| + \sum_{n=1}^{\infty}\left[\max_{0\le t\le T}|u_{cn}^{(N)}(t)| + \max_{0\le t\le T}|u_{sn}^{(N)}(t)|\right] = \|\overline{u}^{(N)}(t)\|_{B_T}. \tag{3.5}$$

First, we will show that $\overline{u}^{(N)}(t) \in B_T$. According to the conditions of the theorem it is easily seen that

$$\begin{aligned}
\|\overline{u}^{(0)}(t)\|_{B_T} &= \frac{1}{2}\max_{0\le t\le T}|u_0^{(0)}(t)| + \sum_{n=1}^{\infty}\left[\max_{0\le t\le T}|u_{cn}^{(0)}(t)| + \max_{0\le t\le T}|u_{sn}^{(0)}(t)\right]\\
&= \frac{1}{2}(|\varphi_0| + |\psi_0|T) + \sum_{k=1}^{\infty}\left[(|\varphi_{ck}| + \frac{1}{\alpha_k}|\psi_{ck}|) + (|\varphi_{sk}| + \frac{1}{\alpha_k}|\psi_{sk}|)\right] < \infty.
\end{aligned}$$

Taking $N=0$ in the equalities (3.4), we obtain

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\left\{f[\tau,\xi,Au^{(0)}(t,\xi)] - f(\tau,\xi,0)\right\} d\xi\, d\tau + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau) f(\tau,\xi,0)\, d\xi\, d\tau.$$

Then applying the Cauchy inequality with respect to t to the both integrals in the right hand side of the last equality we have

$$\begin{aligned}
|u_0^{(1)}(t)| &\le |u_0^{(0)}(t)|\\
&+ \frac{2}{\pi}\left[\int_0^t (t-\tau)^2\, d\tau\right]^{1/2}\left(\int_0^t\left[\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(t,\xi)] - f(\tau,\xi,0)\right\} d\xi\right]^2 d\tau\right)^{1/2}\\
&+ \frac{2}{\pi}\left[\int_0^t (t-\tau)^2\, d\tau\right]^{1/2}\left(\int_0^t\left[\int_0^\pi f(\tau,\xi,0)\ \ d\xi\right]^2 d\tau\right)^{1/2}.
\end{aligned}$$

In the right hand side, we calculate first integrals in the second and the third sums. Than taking first factor is 1 in the second integrals with respect to ξ, again applying Cauchy inequality, and doing some operations, we have

$$|u_0^{(1)}(t)| \le |u_0^{(0)}(t)| + \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\Big(\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\right\}^2 d\xi\, d\tau\Big)^{1/2} + \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\Big(\int_0^t \left[\int_0^\pi f(\tau,\xi,0)\ d\xi\right]^2 d\tau\Big)^{1/2}.$$

Applying Lipschitz condition to the first integral in the right hand side and doing some operation we get

$$|u_0^{(1)}(t)| \le |u_0^{(0)}(t)| + \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\left[\Big(\int_0^t\int_0^\pi b^2(\tau,\xi)\left[Au^{(0)}(\tau,\xi)\right]^2 d\xi\, d\tau\Big)^{1/2} + \|f(\tau,x,0)\|_{L_2(D)}\right],$$

hence we have

$$|u_0^{(1)}(t)| \le |u_0^{(0)}(t)| + \frac{2}{\pi}\sqrt{\frac{\pi T^3}{3}}\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\right]. \quad (3.6)$$

The second equality from (3.4) for $N = 0$ is written as

$$u_{ck}^{(1)}(t) = u_{ck}^{(0)}(t) + \frac{1}{\alpha_k}\frac{2}{\pi}\int_0^t\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(t,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi \sin\alpha_k(t-\tau)\ d\xi\, d\tau + \frac{1}{\alpha_k}\frac{2}{\pi}\int_0^t\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\sin\alpha_k(t-\tau)\ d\xi\, d\tau,$$

Then applying the Cauchy inequality with respect to t to the both integrals in the right hand side we get

$$|u_{ck}^{(1)}(t)| \le |u_{ck}^{(0)}(t)| + \sqrt{T}\frac{1}{\alpha_k}\Big(\int_0^t \frac{2}{\pi}\left[\int_0^\pi \left\{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\right\}\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2} + \sqrt{T}\frac{1}{\alpha_k}\Big(\int_0^t \frac{2}{\pi}\left[\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2}.$$

Taking sum of both sides with respect to $k\,(k = \overline{1,\infty})$, we have

$$\sum_{k=1}^{\infty}|u_{ck}^{(1)}(t)| \le \sum_{k=1}^{\infty}|u_{ck}^{(0)}(t)|$$

$$+\sqrt{T}\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\Big(\int_0^t\left[\frac{2}{\pi}\int_0^{\pi}\left\{f[\tau,\xi,Au^{(0)}(\tau,\xi,)]-f(\tau,\xi,0)\right\}\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2}$$
$$+\sqrt{T}\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\Big(\int_0^t\left[\frac{2}{\pi}\int_0^{\pi}f(\tau,\xi,0)\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2}.$$

Applying Hölder inequality to the second and third sums in the right side of above inequality, then using Bessel inequality related to Fourier coefficients, Lipschitz condition and taking the maximum of integrals with respect to t in the right hand side of resulting inequality yields the following:

$$\sum_{k=1}^{\infty}|u_{ck}^{(1)}(t)|$$
$$\leq\sum_{k=1}^{\infty}|u_{ck}^{(0)}(t)|+M\sqrt{T}\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\right], \quad (3.7)$$

where $M=\left(\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}\right)^{1/2}=\frac{1}{4a}\left(\sum_{k=1}^{\infty}\frac{1}{k^4}\right)^{1/2}=\frac{1}{4a}\left(\frac{\pi^4}{90}\right)^{1/2}=\frac{\pi^2}{120\sqrt{10}}$. Doing similar calculations for $u_{sk}^{(1)}(t)$ we get

$$\sum_{k=1}^{\infty}|u_{sk}^{(1)}(t)|$$
$$\leq\sum_{k=1}^{\infty}|u_{sk}^{(0)}(t)|+M\sqrt{T}\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\right]. \quad (3.8)$$

Using the inequalities (3.6), (3.7) and (3.8) in the following form

$$\frac{|u_0^{(1)}(t)|}{2}+\sum_{k=1}^{\infty}\left[|u_{ck}^{(1)}(t)|+|u_{sk}^{(1)}(t)|\right]\leq\frac{|u_0^{(0)}(t)|}{2}+\sum_{k=1}^{\infty}\left[|u_{ck}^{(0)}(t)|+|u_{sk}^{(0)}(t)|\right]$$
$$+\left(\sqrt{\frac{T^3}{3\pi}}+2M\sqrt{T}\right)\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\right],$$

then taking maximum with respect to t, we obtain that

$$\|\overline{u}^{(1)}(t)\|_{B_T}\leq\|\overline{u}^{(0)}(t)\|_{B_T}$$
$$+\left(\sqrt{\frac{T^3}{3\pi}}+2M\sqrt{T}\right)\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\right].$$

Under the conditions of the theorem we have

$$\|\overline{u}^{(1)}(t)\|_{B_T}<\infty.$$

By the principle of mathematical induction, we obtain that

$$\|\overline{u}^{(N)}(t)\|_{B_T} \leq \|\overline{u}^{(0)}(t)\|_{B_T} + \sqrt{\frac{T}{3\pi}}(T + 2\sqrt{6\pi}M)\|b(t,x)\|_{L_2(D)}\|u^{(N-1)}(t)\|_{B_T} + \sqrt{\frac{T}{3\pi}}(T + 2\sqrt{6\pi}M)\|f(t,x,0)\|_{L_2(D)}.$$

Proceeding in the same way, supposing $\|\overline{u}^{(N)}(t)\|_{B_T} < \infty$, it can be shown that

$$\|\overline{u}^{(N+1)}(t)\|_{B_T} < \infty.$$

Therefore we have proven that

$$\overline{u}^{(N+1)}(t) = \left\{\frac{1}{2}u_0^{(N+1)}(t), u_{c1}^{(N+1)}(t), u_{s1}^{(N+1)}(t), \ldots, u_{ck}^{(N+1)}(t), u_{sk}^{(N+1)}(t), \ldots\right\} \in B_T.$$

Now, to show that the successive approximation sequence $\{\overline{u}^{(N)}(t)\}$ is uniformly convergent in B_T, we examine differences respectively,

$$|u_0^{(N+1)}(t) - u_0^{(N)}(t)|,\ |u_{ck}^{(N+1)}(t) - u_{ck}^{(N)}(t)|,\ |u_{sk}^{(N+1)}(t) - u_{sk}^{(N)}(t)|$$

$(N = \overline{0,\infty}, k = \overline{1,\infty})$. For this let us take

$$|u_0^{(1)}(t) - u_0^{(0)}(t)| = |\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)f[\tau,\xi,Au^{(0)}(\tau,\xi)]\,d\xi\,d\tau| \leq |\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\}\,d\xi\,d\tau| + |\frac{2}{\pi}\int_0^t (t-\tau)\int_0^\pi f(\tau,\xi,0)\,d\xi\,d\tau|,$$

then apply Cauchy inequality with respect to t to the integrals in the right hand side:

$$|u_0^{(1)}(t) - u_0^{(0)}(t)| \leq \left[\int_0^t (t-\tau)^2\,d\tau\right]^{1/2}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\}\,d\xi\right]^2 d\tau\right)^{1/2} + \left[\int_0^t (t-\tau)^2\,d\tau\right]^{1/2}\left(\int_0^t\left[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\,d\xi\right]^2 d\tau\right)^{1/2}.$$

In the right hand side, we calculate the first integral and apply Cauchy inequality with respect to ξ to the second integrals in the both parts:

$$|u_0^{(1)}(t) - u_0^{(0)}(t)| \leq 2T\sqrt{\frac{T}{3\pi}}\left[\int_0^t \int_0^\pi \{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\}^2\, d\xi\, d\tau\right]^{1/2} + 2T\sqrt{\frac{T}{3\pi}}\left[\int_0^t \int_0^\pi f^2(\tau,\xi,0)\, d\xi\, d\tau\right]^{1/2}.$$

Applying Lipschitz inequality to the first part of right hand side after some operations, we get

$$|u_0^{(1)}(t) - u_0^{(0)}(t)| \leq 2T\sqrt{\frac{T}{3\pi}}\big(\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T} + \|f(t,x,0)\|_{L_2(D)}\big). \quad (3.9)$$

With similar operation we obtain that

$$|u_{ck}^{(1)}(t) - u_{ck}^{(0)}(t)| \leq \frac{\sqrt{T}}{\alpha_k}\Big(\int_0^t \left[\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\}\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2} + \frac{\sqrt{T}}{\alpha_k}\Big(\int_0^t \left[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2}.$$

Taking the sum with respect to $k\,(k=\overline{1,\infty})$ to the last inequality and applying the Hölder inequality to the both part of the right, we have

$$\sum_{k=1}^\infty |u_{ck}^{(1)}(t) - u_{ck}^{(0)}(t)| \leq \sqrt{T}\Big(\sum_{k=1}^\infty \frac{1}{\alpha_k^2}\Big)^{1/2}\Big(\sum_{k=1}^\infty \int_0^t \left[\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au^{(0)}(t,\xi)] - f(\tau,\xi,0)\}\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2} + \sqrt{T}\Big(\sum_{k=1}^\infty \frac{1}{\alpha_k^2}\Big)^{1/2}\Big(\sum_{k=1}^\infty \int_0^t \left[\frac{2}{\pi}\int_0^\pi f(\tau,\xi,0)\cos 2k\xi\, d\xi\right]^2 d\tau\Big)^{1/2}.$$

According to the assumptions of the theorem, taking into account integrability of the series in the right hand side by term of term, then using Bessel inequality, we obtain

$$\sum_{k=1}^\infty |u_{ck}^{(1)}(t) - u_{ck}^{(0)}(t)| \leq M\sqrt{\frac{2T}{\pi}}\left[\Big(\int_0^t \int_0^\pi \{f[\tau,\xi,Au^{(0)}(\tau,\xi)] - f(\tau,\xi,0)\}^2\, d\xi\, d\tau\Big)^{1/2} + \Big(\int_0^t \int_0^\pi f^2(\tau,\xi,0)\, d\xi\, d\tau\Big)^{1/2}\right].$$

Applying Lipschitz inequality to the first integral in the right hand side and taking the maximum with respect to t we get

$$\sum_{k=1}^{\infty}|u_{ck}^{(1)}(t)-u_{ck}^{(0)}(t)| \le M\sqrt{\frac{2T}{\pi}}\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\right]. \quad (3.10)$$

In the same way, we obtain

$$\sum_{k=1}^{\infty}|u_{sk}^{(1)}(t)-u_{sk}^{(0)}(t)| \le M\sqrt{\frac{2T}{\pi}}\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)_{L_2(D)}\|\right]. \quad (3.11)$$

By the inequalities (3.9), (3.10) and (3.11) we can write

$$\frac{1}{2}|u_0^{(1)}(t)-u_0^{(0)}(t)|+\sum_{k=1}^{\infty}\left[|u_{ck}^{(1)}(t)-u_{ck}^{(0)}(t)|+|u_{sk}^{(1)}(t)-u_{sk}^{(0)}(t)|\right]$$
$$\le (T+2\sqrt{6}M)\sqrt{\frac{T}{3\pi}}\left[\|b(t,x)\|_{L_2(D)}\|\overline{u}^{(0)}(t)\|_{B_T}+\|f(t,x,0)\|_{L_2(D)}\right](=A_T), \quad (3.12)$$

where it is clear that A_T is positive. Taking maximum with respect to t to left part of the last inequality,

$$\|\overline{u}_0^{(1)}(t)-\overline{u}_0^{(0)}(t)\|_{B_T}\le A_T$$

can be written.

From the above by the principle of mathematical induction, we obtain

$$\|\overline{u}_0^{(N+1)}(t)-\overline{u}_0^{(N)}(t)\|_{B_T}\le A_T\left[(T+2\sqrt{6}M)\sqrt{\frac{T}{3\pi}}\right]^N\frac{\|b(t,x)\|_{L_2(D)}^N}{\sqrt{N!}} \quad (3.13)$$

is true ($N=\overline{1,\infty}$). From (3.13), the series $\sum_{n=0}^{\infty}|\overline{u}^{(N+1)}(t)-\overline{u}^{(N)}(t)|$ whose elements are taken from B_T is uniformly convergent. So the successive approximation sequence $\{\overline{u}^{(N+1)}(t)\}$ whose the general term

$$\overline{u}^{(N+1)}(t)=\overline{u}^{(0)}(t)+\sum_{n=1}^{N}\left[\overline{u}^{(n+1)}(t)-\overline{u}^{(n)}(t)\right]$$

is uniformly convergent in B_T.

Let $\lim_{n\to\infty}\overline{u}^{(N+1)}(t) = \overline{u}(t) = \left\{\frac{1}{2}u_0(t), u_{c1}(t), u_{s1}(t), \dots, u_{ck}(t), u_{sk}(t), \dots\right\}$. Proving $\overline{u}(t)$ satisfies (3.2), we put $\overline{u}(t)$ into (3.2) and the absolute value (σ) of the difference of the right part of the system (3.2) and (3.3) can be written as

$$\sigma \le \frac{2}{\pi}\left|\int_0^t\int_0^\pi (t-\tau)\{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\,d\xi\,d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{2}{\pi}\frac{1}{\alpha_k}\left|\int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau\right|$$
$$+\sum_{k=1}^{\infty}\frac{2}{\pi}\frac{1}{\alpha_k}\left|\int_0^t\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\sin 2k\xi\sin\alpha_k(t-\tau)\,d\xi\,d\tau\right|$$
$$\le \sqrt{\frac{T^3}{3}}\Big(\int_0^t \frac{2}{\pi}\left[\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi\,d\xi\right]^2 d\tau\Big)^{1/2}$$
$$+\sqrt{T}\Big(\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\Big)^{1/2}\left[\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi\,d\xi\Big)^2\right]^{1/2}$$
$$+\sqrt{T}\Big(\sum_{k=1}^{\infty}\frac{1}{\alpha_k}\Big)^{1/2}\left[\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\sin 2k\xi\,d\xi\Big)^2\right]^{1/2}.$$

Then, by the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ we obtain

$$\sigma^2 \le T^3\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\,d\xi\Big)^2 d\tau$$
$$+3M^2T\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi\,d\xi\Big)^2 d\tau$$
$$+3M^2T\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\sin 2k\xi\,d\xi\Big)^2 d\tau.$$

Supposing that $\max(2T^3, 3M^2T) = M_T$, then

$$\sigma^2 \le M_T\int_0^t\frac{1}{2}\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\,d\xi\Big)^2 d\tau$$
$$+M_T\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\cos 2k\xi\,d\xi\Big)^2 d\tau$$
$$+M_T\sum_{k=1}^{\infty}\int_0^t\Big(\frac{2}{\pi}\int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}\sin 2k\xi\,d\xi\Big)^2 d\tau.$$

Applying the Bessel inequality on the right hand side and then Lipschitz condition we get

$$\sigma^2 \le \frac{2}{\pi} M_T \int_0^t \int_0^\pi \{f[\tau,\xi,Au(\tau,\xi)] - f[\tau,\xi,Au^{(N)}(\tau,\xi)]\}^2 \, d\xi \Big)^2 \, d\tau$$
$$\le \frac{2}{\pi} M_T \int_0^t \int_0^\pi b^2(\tau,\xi) \left[Au(\tau,\xi) - Au^{(N)}(\tau,\xi)\right]^2 \, d\xi \, d\tau$$
$$\le \frac{2}{\pi} M_T \|b^2(t,x)\|^2_{L_2(D)} \|\overline{u}(t) - \overline{u}^{(N)}(t)\|_{B_T}.$$

Hence, taking account into $\lim_{n\to\infty} \|\overline{u}(t,\varepsilon) - \overline{u}^{(N)}(t,\varepsilon)\| = 0$, the limit of the norm $\|\overline{u}(t,\varepsilon) - \overline{u}^{(N+1)}(t,\varepsilon)\|$ which is formed by the difference of (3.2) and (3.3) equals to zero for $N \to \infty$. It means that $\overline{u}(t)$ is the solution of the system (3.2).

For the uniqueness, by contradiction, we assume that $\overline{v}(t)$ is also solution of the system. Evaluating the difference $|\overline{u}(t) - \overline{v}(t)|$ we get

$$[\overline{u}(t) - \overline{v}(t)]^2 \le \frac{2}{\pi} M_T \int_0^t \Big(\int_0^\pi b^2(\tau,\xi) \, d\xi \Big) [\overline{u}(t) - \overline{v}(t)]^2 \, d\tau.$$

According to Gronwall inequality, we have $|\overline{u}(t) - \overline{v}(t)| \le 0$, means that $\overline{u}(t) = \overline{v}(t)$. The theorem is thus proven. □

By Theorem 3.1, the following theorem related to the weak solution of problem (2.1)-(2.3) is also true.

Theorem 3.2. *Under the assumptions of Theorem 3.1, there is a unique generalized solution of problem (2.1)-(2.3) and this solution can be found as uniformly convergent series (3.1) in $C(D)$.*

Due to the significance in practical application it is useful to examine the difference between exact solution of the system (3.2)

$$\bar{u}(t) = \Big\{\frac{1}{2}u_0(t), u_{c1}(t), u_{s1}(t), \ldots, u_{ck}(t), u_{sk}(t), \ldots\Big\}$$

and

$$\bar{u}^{(N+1)}(t) = \Big\{\frac{1}{2}u_0^{(N+1)}(t), u_{c1}^{(N+1)}(t), u_{s1}^{(N+1)}(t), \ldots, u_{ck}^{(N+1)}(t), u_{sk}^{(N+1)}(t), \ldots\Big\}$$

$(N+1)$-th successive approximation. By the same way, the following theorem is also achieved.

Theorem 3.3. *Under the assumptions of Theorem 3.1, for the difference of exact solution $\bar{u}(t)$ of the system (3.2) and $\bar{u}^{(N+1)}(t)$ successive approximation, the following inequality*

$$\|\overline{u}(t) - \overline{u}^{(N+1)}(t)\|_{B_T} \le \sqrt{\frac{2}{\pi}\frac{M_T}{N!}} \left[T + 2\sqrt{6}M\right]^N \|b(t,x)\|^{N+1}_{L_2(D)} \exp \frac{M_T}{\pi} \|b(t,x)\|_{L_2(D)}$$

is true.

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