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SERIES EXPANSION AND MODIFIED DECOMPOSITION METHODS FOR LANE-EMDEN EQUATIONS OF INDEX k

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Abstract: In this study, Lane-Emden equations of index k are treated using series expansion and modified Adomian decomposition methods. The special structure of this equation has been exploited to obtain two numerically efficient algorithms suitable for computer programming. We will show that the Taylor series of the answer can be found by direct substitution. We will also show that we do not need to compute Adomian's polynomials for these equations.

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1. Introduction

The Lane-Emden equation of index k is a basic equation in the theory of stellar structure, see [13]. The equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics [13], [9], [7]. The Lane-Emden equation of index k is of the form

$$y'' + \frac{2}{x}y' + y^k = 0, \tag{1}$$

which has been the object of much study [9], [7], [12], [5]. The boundary conditions, which are of most interest [9], are the following:

$$y(0) = 1, \qquad y'(0) = 0.$$

In this paper, we will consider the boundary conditions to be

$$y(0) = y_0, \qquad y'(0) = 0.$$

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At the beginning of 1980's Adomian proposed a new method to solve some functional equations [2], [3]. This method and its modifications [10], [14], [15], [16] have been efficiently used to solve singular and nonsingular ODE's. The theoritical treatment of the convergence of ADM has been considered in [1], [4], [6], [8].

In this paper, we are going to utilize series expansion and modified decomposition methods in efficient ways for solving Lane-Emden equations of index k. The major difficulty in using ADM is computing Adomian's polynomials. Here we introduce a scheme which does not need to compute Adomian's polynomials. We will also show that the Taylor series of the answer can be found by direct substitution.

2. Analysis of the Methods

Suppose H is a Hilbert space and consider the following functional equation:

$$y = f + Ny, \tag{2}$$

where $N : H \mapsto H$ is a nonlinear operator on H, f is a given function in H and we are looking for $y \in H$ satisfying (2).

At the beginning of 1980's Adomian developed a very powerful technique for solving (2) where the solution y is considered in the form of the series:

$$y = \sum_{i=1}^{\infty} u_i \tag{3}$$

and Ny is expanded in the form of Adomian series:

$$Ny = \sum_{n=0}^{\infty} A_n. \tag{4}$$

The method consists of the following scheme:

$$\begin{cases} u_0 = f, \\ u_{n+1} = A_n(u_0, u_1, ..., u_n), \end{cases} \tag{5}$$

where each A_n is a polynomial in $u_0, u_1, ..., u_n$ called an Adomian polynomial obtained by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, \qquad n = 0, 1, 2, \dots . \tag{6}$$

The standard ADM usually defines the equation in an operator form by considering the highest-order derivative contained in the problem

$$L = \frac{d^n}{dx^n}. \tag{7}$$

Consider the initial value problem of Lane-Emden equation formulated as

$$\begin{cases} y'' + \frac{2}{x}y' + F(x,y) = g(x), & 0 < x \leq 1, \\ y(0) = A, \quad y'(0) = B, \end{cases} \tag{8}$$

where A and B are given constants, $F(x,y)$ is a given real function and $g(x) \in \mathcal{C}[0,1]$ is given. Usually, the standard ADM may be divergent in solving singular Lane-Emden equations. To overcome the singularity behavior, Wazwaz [15] defined the differential operator L in terms of two derivatives contained in the problem. He wrote (8) in the form

$$L(y) = -F(x,y) + g(x), \tag{9}$$

where the differential operator L is defined by

$$L = x^{-2}\frac{d}{dx}(x^2\frac{d}{dx}). \tag{10}$$

We must mention that even when both schemes are convergent, the scheme proposed by Waswas is usually faster [11].

2.1. Modified Decomposition Method

If we take $L = x^{-2}\frac{d}{dx}(x^2\frac{d}{dx})$, equation (1) becomes

$$Ly = -y^k. \tag{11}$$

Now we set $y = \sum_{n=0}^{\infty} u_n$ and $y^k = \sum_{n=0}^{\infty} A_n$ and in an obvious way by putting

$$\begin{cases} u_0 = y_0, \\ u_n = -\int_0^x x^{-2}\int_0^x x^2 A_{n-1}dxdx, \quad n = 1,2,\dots. \end{cases} \tag{12}$$

we can find u_n's. To obtain A_n's we exploit the following fact: In the expansion of $(u_0+u_1+u_2+\dots)^k$ we have terms in the general form $\frac{k!}{k_1!k_2!\dots k_l!}u_{i_1}^{k_1}u_{i_2}^{k_2}\dots u_{i_l}^{k_l}$, where k_i's are non-negative integers satisfying $k_1 + k_2 + \dots + k_l = k$. Since A_0 should depend only on u_0, the only term that appears in A_0 is u_0^k whose coefficient is $\frac{k!}{k!} = 1$. Now u_1 comes into account and the terms appearing in A_1 are $u_0^{k-1}u_1$, $u_0^{k-2}u_1^2$,..., $u_0u_1^{k-1}$, u_1^k whose coefficients are $\frac{k!}{(k-1)!1!}$, $\frac{k!}{(k-2)!2!}$, ..., $\frac{k!}{1!(k-1)!}$, $\frac{k!}{k!}$, respectively, and so on. Note that these A_i's are different from Adomian's polynomials because if we use the formula (6) we obtain

$$A_0 = u_0^k, \quad A_1 = ku_1u_0^{k-1}, \quad A_2 = ku_0^{k-1}u_2 + \frac{k(k-1)}{2}u_0^{k-2}u_1^2, \dots. \tag{13}$$

Here the term $u_0^{k-2}u_1^2$ appears in A_2 but we obtained it in A_1.

2.2. Series Expansion Method

If we want to substitute $y = \sum_{n=0}^{\infty} a_n x^n$ in $y'' + \frac{2}{x}y' + y^k = 0$ and compute a_i's, the only thing we need is to expand $y^k = (a_0 + a_1x + a_2x^2 + ...)^k$ and order it by powers of x obtaining $y^k = b_0 + b_1x + b_2x^2 + ...$, where b_i's depend on a_i's. Again we exploit the fact that in the expansion of $(u_0 + u_1 + u_2 + ...)^k$ we have terms in the general form $\frac{k!}{k_1!k_2!...k_l!}u_{i_1}^{k_1}u_{i_2}^{k_2}...u_{i_l}^{k_l}$, where k_i's are non-negative integers satisfying $k_1 + k_2 + ... + k_l = k$. So we have $y^k = (u_0 + u_1 + u_2 + ...)^k$, where $u_0 = a_0$, $u_1 = a_1x$, $u_2 = a_2x^2$, ... The only term containing x^0 is u_0^k whose coefficient is $\frac{k!}{k!}$ so we have $b_0 = a_0^k$. The term containing x^1 is $u_0^{k-1}u_1$ whose coefficient is $\frac{k!}{1!(k-1)!}$ so we have $b_1 = ka_0^{k-1}a_1$. The terms containing x^2 are $u_0^{k-1}u_2$ and $u_0^{k-2}u_1^2$ (assuming $k \geq 2$) with coefficients $\frac{k!}{(k-1)!1!}$ and $\frac{k!}{(k-2)!2!}$, respectively. So we have $b_2 = ka_0^{k-1}a_2 + \frac{k(k-1)}{2}a_0^{k-2}a_1^2$ and so on. Note that even if $k = 1$ and we assume the existence of $u_0^{k-2}u_1^2$, then its coefficient automatically becomes zero and we will obtain $b_2 = a_2$. This fact simplifies the computer programming.

3. Examples

In this section we give a couple of examples to demonstrate the power of the proposed algorithms.

Example 1. Consider the initial value problem

$$\begin{cases} y'' + \frac{2}{x}y' + y^2 = 0, \\ y(0) = 1. \end{cases} \tag{14}$$

Proposed Modified Decomposition Method. Implementing the method described in Section 2.1. we obtain

$$(u_0 + u_1 + u_2 + ...)^2 = A_0 + A_1 + A_2 + ...,$$

where

$$A_0 = u_0^2,$$
$$A_1 = u_1^2 + 2u_0u_1,$$
$$A_2 = u_2^2 + 2u_2u_0 + 2u_2u_1,$$
$$\vdots$$

The above equations together with the recursive relations:

$$\begin{cases} u_0 = 1, \\ u_n = -\int_0^x x^{-2} \int_0^x x^2 A_{n-1} dx dx, \quad n \in \mathbb{N}, \end{cases}$$

imply

$$A_0 = 1$$
$$u_1 = -\frac{1}{6}x^2,$$
$$A_1 = -\frac{1}{3}x^2 + \frac{1}{36}x^4,$$
$$u_2 = \frac{1}{60}x^4 - \frac{1}{1512}x^6,$$
$$A_2 = \frac{1}{30}x^4 - \frac{13}{1890}x^6 + \frac{113}{226800}x^8 - \frac{1}{45360}x^{10} + \frac{1}{2286144}x^{12},$$
$$u_3 = -\frac{1}{1260}x^6 + \frac{13}{136080}x^8 - \frac{113}{24948000}x^{10} + \frac{1}{7076160}x^{12} - \frac{1}{480090240}x^{14}.$$

Accepting $y^* = u_0 + u_1 + u_2 + u_3$ as the approximate solution and substituting it in $N[y] = y'' + \frac{2}{x}y' + y^2$, we obtain the local truncation error to be

$$N[y^*] = -.001587301587x^6 + o(x^8)$$

which is of order x^6. Note that we have done only three iterations.

Series Expansion Method. We assume the answer to be of the form $y = a_0+a_1x+a_2x^2+....$ So we have $y' = a_1+2a_2x+3a_3x^2+...,$ $y'' = 2a_2+6x+12a_4x^2+...$ and implementing the method described in Section 2.2. we obtain

$$y^2 = (a_0 + a_1x + a_2x^2 + ...)^2 = a_0^2 + 2a_0a_1x + (a_1^2 + 2a_0a_2)x^2 + \dots .$$

Substituting in (14) we have

$$2a_2+6a_3x+12a_4x^2+\frac{2a_1}{x}+4a_2+6a_3x+8a_4x^2+a_0^2+2a_0a_1x+(a_1^2+2a_0a_2)x^2+... = 0$$

and by putting the coefficients of x^i equal to zero we obtain

$$a_1 = 0,$$
$$a_2 = -\frac{a_0^2}{6},$$
$$a_3 = 0,$$
$$a_4 = \frac{a_0^3}{60},$$
$$\vdots$$

So we have

$$y = a_0 - \frac{a_0^2}{6}x^2 + \frac{a_0^3}{60}x^4 + \dots .$$

Using the initial condition in (14), i.e. $y(0) = 1$, we obtain $a_0 = 1$. Therefore we obtain the Taylor polynomial of the solution to be

$$y = 1 - \frac{1}{6}x^2 + \frac{1}{60}x^4 + \dots .$$

Note that the obtained three terms and the first three terms we obtained using proposed modified decomposition method are the same. If we substitute $y^* = 1 + \frac{1}{6}x^2 + \frac{1}{60}x^4$ in $N(y) = y'' + \frac{2}{x}y' + y^2$, we obtain the local truncation error to be

$$N(y^*) = -\frac{11}{180}x^4 - \frac{1}{180}x^6 + \frac{1}{3600}x^8.$$

Example 2. Consider the initial value problem

$$\begin{cases} y'' + \dfrac{2}{x}y' + y^7 = 0, \\ y(0) = -1. \end{cases} \tag{15}$$

Proposed Modified Decomposition Method. Implementing the method described in Section 2.1. we obtain

$$(u_0 + u_1 + u_2 + ...)^7 = A_0 + A_1 + ...,$$

where

$$A_0 = u_0^7,$$

$$A_1 = u_1^7 + 7u_0u_1^6 + 21u_0^2u_1^5 + 35u_0^3u_1^4 + 35u_0^4u_1^3 + 21u_0^5u_1^2 + 7u_0^6u_1.$$

The above equations together with the recursive relations:

$$\begin{cases} u_0 = -1, \\ u_n = -\int_0^x x^{-2}\int_0^x x^2 A_{n-1}dxdx, \qquad n \in \mathbb{N}, \end{cases}$$

imply

$$A_0 = -1,$$

$$u_1 = \frac{1}{6}x^2,$$

$$A_1 = \frac{7}{6}x^2 - \frac{7}{12}x^4 + \frac{35}{216}x^6 - \frac{35}{1296}x^8 + \frac{7}{2592}x^{10} - \frac{1}{46656}x^{14},$$

$$u_2 = -\frac{7}{120}x^4 + \frac{1}{72}x^6 - \frac{35}{15552}x^8 + \frac{7}{28512}x^{10} - \frac{7}{404352}x^{12} + \frac{1}{12690432}x^{16}.$$

Accepting $y^* = u_0 + u_1 + u_2$ as the approximate solution and substituting it in $N[y] = y'' + \frac{2}{x}y' + y^7$, we obtain the local truncation error to be

$$N[y^*] = -\frac{49}{120}x^4 + o(x^6)$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

which is of order x^4. Note that we have done only two iterations.

Series Expansion Method. We assume the answer to be of the form $y = a_0+a_1x+a_2x^2+\dots$. So we have $y' = a_1+2a_2x+3a_3x^2+\dots$, $y'' = 2a_2+6x+12a_4x^2+\dots$ and implementing the method described in Section 2.2. we obtain

$$y^7 = (a_0 + a_1x + a_2x^2 + \dots)^7 = a_0^7 + 7a_0^6a_1x + (21a_0^5a_1^2 + 7a_0^6a_2)x^2 + \dots .$$

Substituting in (15) we have

$$2a_2+6a_3x+12a_4x^2+\frac{2a_1}{x}+4a_2+6a_3x+8a_4x^2+a_0^7+7a_0^6a_1x+(21a_0^5a_1^2+7a_0^6a_2)x^2+\dots = 0$$

and by putting the coefficients of x^i equal to zero we obtain

$$a_1 = 0,$$

$$a_2 = -\frac{a_0^7}{6},$$

$$a_3 = 0,$$

$$a_4 = \frac{7a_0^{13}}{120},$$

$$\vdots$$

So we have

$$y = a_0 - \frac{a_0^7}{6}x^2 + \frac{7a_0^{13}}{120}x^4 + \dots .$$

Using the initial condition in (15), i.e. $y(0) = -1$, we obtain $a_0 = -1$. Therefore we obtain the Taylor series of the solution to be

$$y = -1 + \frac{1}{6}x^2 - \frac{7}{120}x^4 + \dots .$$

Note that the obtained three terms and the first three terms we obtained using proposed modified decomposition method are the same. If we substitute $y^* = -1 + \frac{1}{6}x^2 - \frac{7}{120}x^4$ in $N(y) = y'' + \frac{2}{x}y' + y^7$, we obtain the local truncation error to be

$$N(y^*) = -\frac{119}{120}x^4 + o(x^6).$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

4. Conclusion

Despite the fact that Lane-Emden equations of index k are non-linear, Taylor series of their solution can be found by direct substitution. In order to do so, we exploited these equations' special structure. We also exploited this special structure in order to avoid computing Adomian polynomials while implementing a modified decomposition method.

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