International Electronic Journal of Pure and Applied Mathematics

Volume 1 No. 1 2010, 29-36

A FIXED POINTS THEOREM ON THREE COMPLETE METRIC SPACES

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Abstract: A fixed point theorem for three mappings on three metric spaces is proved. This result is a modification of the result of Nešić [2] from two mappings of a metric space into itself, to three mappings of different metric spaces. We have modified the methods used by Nešić [2] and by Jain, Shrivastava and Fischer [1]. We also show that the theorem of Nung [3] is a corollary of our result and that the continuity of only one of the mappings is sufficient.

AMS Subject Classification: 47H10, 54H25
Key Words: fixed point, metric space, complete metric space

1. Introduction

In [2], the following theorem is proved.

Theorem 1.1. *Let (X, d) be a metric space and S, T be two mappings of X into itself, satisfying the following inequality:*

$$[1 + pd(x, y)]d(Sx, Ty) \le p[d(x, Sx)d(y, Ty) + d(x, Ty)d(y, Sx)] + q \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\},$$

for all $x, y \in X$, where $p \ge 0$ and $0 \le q < 1$.

If (X, d) is (S, T)-orbitally complete metric space, then S and T have a unique common fixed point u in X.

Received: March 5, 2010

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In [3], the following theorem is proved.

Theorem 1.2. *Let* $(X, d_1), (Y, d_2), (Z, d_3)$ *be three complete metric spaces and* T *be a continuous mapping of* X *into* Y, S *a continuous mapping of* Y *into* Z *and* R *be a continuous of* Z *into* X, *satisfying the following inequalities:*

$$\begin{aligned} d_1(RSTx, RSy) &\leq c\max\{d_1(x, RSy), d_1(x, RSTx), d_2(y, Tx), d_3(Sy, STx)\}, \\ d_2(TRSy, TRz) &\leq c\max\{d_2(y, TRz), d_2(y, TRSy), d_3(Z, Sy), d_1(Rz, RSy)\}, \\ d_3(STRz, STx) &\leq c\max\{d_3(z, STx), d_3(z, STRz), d_1(x, Rz), d_2(Tx, TRz)\}, \end{aligned}$$

for all $x \in X, y \in Y$ *and* $z \in Z$, *where* $0 \leq c < 1$. *Then* RST *has an unique fixed point* $u \in X$, TRS *has an unique fixed point* $v \in Y$ *and* STR *has an unique fixed point* $w \in Z$. *Further,* $Tu = v, Sv = w$ *and* $Rw = u$.

In this paper we will give a generalization of Theorem 1.2 modifying the results of Nešić [2]. We will also show that in Theorem 1.2 it is not nescessary the continuity of the three mappings, but it is sufficient the continuity of only one of them.

2. Main Results

Theorem 2.1. *Let* $(X, d_1), (Y, d_2), (Z, d_3)$ *be three complete metric spaces and* $T : X \to Y, S : Y \to Z$ *and* $R : Z \to X$ *be three mappings from which at least one of them is continuous, satisfying the following inequality:*

$$\begin{aligned} &[1 + pd_1(x, RSy) + pd_2(y, Tx)]d_1(RSy, RSTx) \\ &\leq p[d_1(x, RSy)d_3(Sy, STx) + d_1(x, RSTx)d_2(y, TRSy) + d_1(x, RSy)d_2(y, Tx)] \\ &\qquad + q\max\{d_1(x, RSy), d_1(x, RSTx), d_2(y, Tx), d_3(STx, Sy)\}, \end{aligned} \tag{1}$$

$$\begin{aligned} &[1 + pd_2(y, TRz) + pd_3(z, Sy)]d_2(TRz, TRSy) \\ &\leq p[d_2(y, TRz)d_1(Rz, RSy) + d_2(y, TRSy)d_3(z, STRz) + d_2(y, TRz)d_3(z, Sy)] \\ &\qquad + q\max\{d_2(y, TRz), d_2(y, TRSy), d_3(z, Sy), d_1(RSy, Rz)\}, \end{aligned} \tag{2}$$

$$\begin{aligned} &[1 + pd_3(z, STx) + pd_1(x, Rz)]d_3(STx, STRz) \\ &\leq p[d_3(z, STx)d_2(Tx, TRz) + d_3(z, STRz)d_1(x, RSTx) + d_3(z, STx)d_1(x, Rz)] \\ &\qquad + q\max\{d_3(z, STx), d_3(z, STRz), d_1(x, Rz), d_2(TRz, Tx)\}, \end{aligned} \tag{3}$$

for all $x \in X, y \in Y, z \in Z$, *where* $p \geq 0$ *and* $0 \leq q < 1$. *Then* RST *has an unique fixed point* $\alpha \in X$, TRS *has an unique fixed point* $\beta \in Y$ *and* STR *has an unique fixed point* $\gamma \in Z$. *Further,* $T\alpha = \beta, S\beta = \gamma$ *and* $R\gamma = \alpha$.

Proof. Let $x_0 \in X$ be an arbitrary point. We define the sequences $(x_n), (y_n)$ and (z_n) in X, Y and Z respectively as follows:

$$x_n = (RST)^n x_0, \quad y_n = Tx_{n-1}, \quad z_n = Sy_n,$$

for $n = 1, 2, \ldots$.

By the inequality (2), for $y = y_n$ and $z = z_{n-1}$ we get:

$$\begin{aligned}&[1 + pd_2(y_n, y_n) + pd_3(z_{n-1}, z_n)]d_2(y_n, y_{n+1}) \\ &\quad \leq p[d_2(y_n, y_n)d_1(x_{n-1}, x_n) + d_2(y_n, y_{n+1})d_3(z_{n-1}, z_n) + d_2(y_n, y_n)d_3(z_{n-1}, z_n)] \\ &\qquad + q \max\{d_2(y_n, y_n), d_2(y_n, y_{n+1}), d_3(z_{n-1}, z_n), d_1(x_n, x_{n-1})\},\end{aligned}$$

from which it follows:

$$d_2(y_n, y_{n+1}) \leq q \max\{d_2(y_n, y_{n+1}), d_1(x_n, x_{n-1}), d_3(z_n, z_{n-1})\} = q \max A,$$

where $A = \{d_2(y_n, y_{n+1}), d_1(x_n, x_{n-1}), d_3(z_n, z_{n-1})\}$.

If $\max A = d_2(y_n, y_{n+1})$, then we have:

$$d_2(y_n, y_{n+1}) \leq qd_2(y_n, y_{n+1})$$

and since $0 \leq q < 1$, it follows $d_2(y_n, y_{n+1}) = 0$. Thus we have:

$$d_2(y_n, y_{n+1}) \leq q \max\{d_1(x_n, x_{n-1}), d_3(z_n, z_{n-1})\}. \tag{4}$$

In the same way, by (3), for $x = x_{n-1}$ and $z = z_n$, we get:

$$\begin{aligned}&[1 + pd_3(z_n, z_n) + pd_1(x_{n-1}, x_n)]d_3(z_n, z_{n+1}) \\ &\quad \leq p[d_3(z_n, z_n)d_2(y_n, y_{n+1}) + d_3(z_n, z_{n+1})d_1(x_{n-1}, x_n) + d_3(z_n, z_n)d_1(x_n, x_n)] \\ &\qquad + q \max\{d_3(z_n, z_n), d_3(z_n, z_{n+1}), d_1(x_{n-1}, x_n), d_2(y_{n+1}, y_n)\},\end{aligned}$$

from which we get:

$$d_3(z_n, z_{n+1}) \leq q \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n)\}. \tag{5}$$

In the same way, by (1), for $y = y_n$ and $x = x_n$ we get:

$$\begin{aligned}&[1 + pd_1(x_n, x_n) + pd_2(y_n, y_{n+1})]d_1(x_n, x_{n+1}) \\ &\quad \leq p[d_1(x_n, x_n)d_3(z_n, z_{n+1}) + d_1(x_n, x_{n+1})d_2(y_n, y_{n+1}) + d_1(x_n, x_n)d_2(y_n, y_{n+1})] \\ &\qquad + q \max\{d_1(x_n, x_n), d_1(x_n, x_{n+1}), d_2(y_n, y_{n+1}), d_3(z_{n+1}, z_n)\},\end{aligned}$$

from which we get:

$$d_1(x_n, x_{n+1}) \leq q \max\{d_1(x_n, x_{n+1}), d_2(y_n, y_{n+1}), d_3(z_n, z_{n+1})\}$$

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and by (4) and (5) we have:

$$d_1(x_n, x_{n+1}) \leq q \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n)\}. \tag{6}$$

Taking n equal with $n-1, n-2, ...,$ using (4), (5) and (6) we obtain:

$$\begin{aligned} d_1(x_n, x_{n+1}) &\leq q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}, \\ d_2(y_n, y_{n+1}) &\leq q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}, \\ d_3(z_n, z_{n+1}) &\leq q^{n-1} \max\{d_1(x_1, x_2), d_3(z_1, z_2)\}. \end{aligned}$$

Since $0 \leq q < 1$, the sequences $(x_n), (y_n)$ and (z_n) are Cauchy sequences with limit α, β and γ in X, Y and Y, respectively.

Suppose that the mapping S is continuous. Then by

$$\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n,$$

we get:

$$S\beta = \gamma. \tag{7}$$

By (1), for $y = \beta$ and $x = x_n$ we get:

$$\begin{aligned} &[1 + pd_1(x_n, RS\beta) + pd_2(\beta, y_{n+1})]d_1(RS\beta, x_{n+1}) \\ &\leq p[d_1(x_n, RS\beta)d_3(S\beta, z_{n+1}) + d_1(x_n, x_{n+1})d_2(\beta, TRS\beta) + d_1(x_n, RS\beta)d_2(\beta, y_{n+1})] \\ &\qquad + q \max\{d_1(x_n, RS\beta), d_1(x_n, x_{n+1}), d_2(\beta, y_{n+1}), d_3(\gamma, S\beta)\} \end{aligned}$$

Letting n tend to infinity, by the fact that $S\beta = \gamma$ we get:

$$\begin{aligned} [1 + pd_1(\alpha, RS\beta)]d_1(\alpha, RS\beta) &\leq qd_1(\alpha, RS\beta), \\ d_1(\alpha, RS\beta) &\leq \frac{q}{1 + pd_1(\alpha, RS\beta)} d_1(\alpha, RS\beta), \end{aligned}$$

from which it follows:

$$d_1(\alpha, RS\beta) = 0 \Leftrightarrow RS\beta = \alpha, \tag{8}$$

since

$$\frac{q}{1 + pd_1(\alpha, RS\beta)} \leq q < 1.$$

By (2), for $z = S\beta$ and $y = y_n$, we get:

$$\begin{aligned} &[1 + pd_2(y_n, TRS\beta) + pd_3(S\beta, z_n)]d_2(TRS\beta, y_{n+1}) \\ &\leq p[d_2(y_n, TRS\beta)d_1(RS\beta, x_n) \\ &\quad + d_2(y_n, y_{n+1})d_3(S\beta, STRS\beta) + d_2(y_n, TRS\beta)d_3(S\beta, z_n)] \\ &\quad + q \max\{d_2(y_n, TRS\beta), d_2(y_n, y_{n+1}), d_3(S\beta, z_n), d_1(x_n, RS\beta)\}. \end{aligned}$$

Letting n tend to infinity, by (7) and (8) we get:

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$$[1 + pd_2(\beta, TRS\beta)]d_2(TRS\beta, \beta) \leq qd_2(TRS\beta, \beta)$$

from which it follows $d_2(TRS\beta, \beta) = 0$, or

$$TRS\beta = \beta. \tag{9}$$

By (7), (8), (9) we get:

$$\begin{aligned} TRS\beta &= TR\gamma = T\alpha = \beta, \\ STR\gamma &= ST\alpha = S\beta = \gamma, \\ RST\alpha &= RS\beta = R\gamma = \alpha. \end{aligned}$$

Thus, we proved that the points α, β, γ are fixed points of RST, TRS and STR respectively.

In the same conclusion we would arrive if one of the mappings R or T would be continuous.

We emphasize the fact that it is sufficient the contiunity of only one of the mappings T, S and R.

Let we prove now the unicity of the fixed points α, β and γ.

Assume that there is α' a fixed point of RST different from α. By (1), for $y = T\alpha$ and $x = \alpha'$, we get:

$$\begin{aligned} &[1 + pd_1(\alpha', RST\alpha) + pd_2(T\alpha, T\alpha')]d_1(RST\alpha, RST\alpha') \\ &\leq p[d_1(\alpha', RST\alpha)d_3(ST\alpha, ST\alpha') \\ &+ d_1(\alpha', RST\alpha')d_2(T\alpha, TRST\alpha) + d_1(\alpha', RST\alpha)d_2(T\alpha, T\alpha')] \\ &+ q\max\{d_1(\alpha', RST\alpha), d_1(\alpha', RST\alpha'), d_2(T\alpha, T\alpha'), d_3(ST\alpha', ST\alpha)\}, \end{aligned}$$

or

$$\begin{aligned} &[1 + pd_1(\alpha', \alpha) + pd_2(T\alpha, T\alpha')]d_1(\alpha, \alpha') \\ &\leq p[d_1(\alpha', \alpha)d_3(ST\alpha, ST\alpha') + 0 + d_1(\alpha', \alpha)d_2(T\alpha, T\alpha')] \\ &+ q\max\{d_1(\alpha', \alpha), 0, d_2(T\alpha, T\alpha'), d_3(ST\alpha', ST\alpha)\}, \end{aligned}$$

or

$$\begin{aligned} [1 + pd_1(\alpha, \alpha')]d_1(\alpha, \alpha') &\leq p[d_1(\alpha, \alpha')d_3(ST\alpha, ST\alpha') \\ &+ q\max\{d_1(\alpha, \alpha'), d_2(T\alpha, T\alpha'), d_3(ST\alpha, ST\alpha')\}. \end{aligned} \tag{10}$$

In respect of $\max\{d_1(\alpha, \alpha'), d_2(T\alpha, T\alpha'), d_3(ST\alpha, ST\alpha')\} = \max A$ we distinguish the following three cases:

Case 1. If $\max A = d_1(\alpha, \alpha')$, we have $d_3(ST\alpha, ST\alpha') \leq d_1(\alpha, \alpha')$, and by (10) we obtain:

$$[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha') + qd_1(\alpha,\alpha') \le \\ \le pd_1(\alpha,\alpha')d_1(\alpha,\alpha') + qd_1(\alpha,\alpha').$$

By the above we obtain $d_1(\alpha,\alpha') \le qd_1(\alpha,\alpha')$ and since $0 \le q < 1$ we get:

$$\alpha = \alpha'. \tag{11}$$

Case 2. If $\max A = d_2(T\alpha,T\alpha')$, we have $d_3(ST\alpha,ST\alpha') \le d_2(T\alpha,T\alpha')$, and by (10) we obtain:

$$[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha') + qd_2(T\alpha,T\alpha') \\ \le pd_1(\alpha,\alpha')d_2(T\alpha,T\alpha') + qd_2(T\alpha,T\alpha'),$$

or

$$\begin{aligned} [1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') &\le [q+pd_1(\alpha,\alpha')]d_2(T\alpha,T\alpha'), \\ d_1(\alpha,\alpha') &\le \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')}d_2(T\alpha,T\alpha'), \end{aligned}$$

from which it follows:

$$d_1(\alpha,\alpha') \le rd_2(T\alpha,T\alpha'), \tag{12}$$

where

$$0 \le r = \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')} < 1,$$

since $0 \le q < 1$.

Case 3. If $\max A = d_3(ST\alpha, St\alpha')$, then the inequality (10) takes the form:

$$[1+pd_1(\alpha,\alpha')]d_1(\alpha,\alpha') \le p[d_1(\alpha,\alpha')d_3(ST\alpha,ST\alpha') + qd_3(ST\alpha,ST\alpha') \\ \le d_1(\alpha,\alpha') \le \frac{q+pd_1(\alpha,\alpha')}{1+pd_1(\alpha,\alpha')}d_3(ST\alpha,ST\alpha') \\ d_1(\alpha,\alpha') \le rd_3(ST\alpha,ST\alpha'). \tag{13}$$

Continuing our argumentation for Case 2, by (2) for $z = ST\alpha$ and $y = T\alpha'$ we have:

$$[1+pd_2(T\alpha',TRST\alpha) + pd_3(ST\alpha,ST\alpha')]d_2(TRST\alpha,TRST\alpha') \\ \le p[d_2(T\alpha',TRST\alpha)d_1(RST\alpha,RST\alpha') + d_2(T\alpha',TRST\alpha')d_3(ST\alpha,STRST\alpha) \\ + d_2(T\alpha',TRST\alpha)d_3(ST\alpha,ST\alpha')] + q\max\{d_2(T\alpha',TRST\alpha), \\ d_2(T\alpha',TRST\alpha'), d_3(ST\alpha,ST\alpha'), d_1(RST\alpha',RST\alpha)\},$$

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or

$$\begin{aligned}[1 + pd_2(T\alpha', T\alpha) + pd_3(ST\alpha, ST\alpha')]d_2(T\alpha, T\alpha') \\ \leq p[d_2(T\alpha', T\alpha)d_1(\alpha, \alpha') + d_2(T\alpha', T\alpha')d_3(ST\alpha, ST\alpha) \\ + d_2(T\alpha', T\alpha)d_3(ST\alpha, ST\alpha')] + q\max\{d_2(T\alpha', T\alpha), \\ d_2(T\alpha', T\alpha'), d_3(ST\alpha, ST\alpha'), d_1(\alpha', \alpha)\},\end{aligned}$$

or

$$\begin{aligned}[1 + pd_2(T\alpha', T\alpha)]d_2(T\alpha, T\alpha')] \leq pd_2(T\alpha', T\alpha)d_1(\alpha, \alpha') \\ + q\max\{d_1(\alpha', \alpha), d_2(T\alpha', T\alpha), d_3(ST\alpha, ST\alpha')\},\end{aligned}$$

or

$$[1 + pd_2(T\alpha', T\alpha)]d_2(T\alpha, T\alpha') \leq pd_2(T\alpha', T\alpha)d_1(\alpha, \alpha') + q\max A. \tag{14}$$

In Case 2, we have $\max A = d_2(T\alpha, T\alpha')$ and by (14) we obtain:

$$[1 + pd_2(T\alpha', T\alpha)]d_2(T\alpha, T\alpha') \leq pd_2(T\alpha', T\alpha)d_2(T\alpha', T\alpha) + qd_2(T\alpha, T\alpha'),$$

or

$$d_2(T\alpha, T\alpha') \leq qd_2(T\alpha, T\alpha').$$

Since $0 \leq q < 1$, we obtain:

$$d_2(T\alpha, T\alpha') = 0$$

and by (13) it follows that $d_1(\alpha, \alpha') = 0$, so we obtain again the inequality (11).

In Case 3, by (3) for $x = RST\alpha, z = ST\alpha'$ and in the same way we obtain:

$$[1 + pd_3(ST\alpha', ST\alpha)]d_3(ST\alpha, ST\alpha') \leq pd_3(ST\alpha', ST\alpha)d_2(T\alpha', T\alpha) + q\max A.$$

Since in this case $\max A = d_3(ST\alpha, ST\alpha')$, we have $d_2(T\alpha', T\alpha) \leq d_3(ST\alpha, ST\alpha')$ and we obtain:

$$\begin{aligned}[1 + pd_3(ST\alpha, ST\alpha')]d_3(ST\alpha, ST\alpha') \\ \leq pd_3(ST\alpha, ST\alpha')d_3(ST\alpha, ST\alpha') + qd_3(ST\alpha, ST\alpha'),\end{aligned}$$

from which it follows

$$d_3(ST\alpha, ST\alpha') \leq qd_3(ST\alpha, ST\alpha').$$

Since $0 \leq q < 1$ we take:

$$d_3(ST\alpha, ST\alpha') = 0,$$

and by (13) it follows $d_1(\alpha, \alpha') = 0$. Thus, again, in this case the following equality holds:

$$\alpha = \alpha'.$$

In the same way, it is proved the unicity of β and γ. □

Corollary 2.2. *Theorem 1.2 in [3] follows from Theorem 2.1 if $p = 0$. Further, it is sufficient the continuity of only one of the three mappings.*

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