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TWO-DIMENSIONAL INTEGRAL INEQUALITIES AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS WITH "MAXIMA"

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Abstract: This paper deals with integral inequalities that involve the maximum of the unknown scalar function of two variables. The considered inequalities are generalizations of the classical integral inequality of Bihari. The importance of these integral inequalities is defined by their wide applications in the qualitative investigations of various properties of solutions of partial differential equations with "maxima" and it is illustrated by some direct applications.

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1. Introduction

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity and are adequately modeled by differential equations with "maxima" (see [16]). The qualitative investigation of properties of differential equations with "maxima" requires building of an appropriate mathematical apparatus, which is developed by D.D. Bainov et al (see [2], [3], [4], [5], [6] , [7], [9], [11], [12]). One of the main mathematical tools, employed successfully for studying existence, uniqueness, continuous dependence, comparison, perturbation, boundedness, and stability of

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solutions of differential and integral equations is the method of integral inequalities. Note that different types of integral inequalities are solved in the book of D. Bainov et al (see [8]) as well as in the papers [1], [10], [13], [14], [17], [18], [19], [20]. The development of the theory of partial differential equations with "maxima" (see [15]) requires solving of two-dimensional integral inequalities that involve maximum of the unknown function.

This paper deals with some linear and nonlinear two-dimensional inequalities. Some of the solved inequalities are applied to partial differential equations with "maxima" to obtain bounds for their solutions.

2. Main Results

Let $h > 0$ be a constant, $x_0,\ y_0,\ X,\ Y$ be fixed such that $0 \le x_0 < X \le \infty$ and $0 \le y_0 < Y \le \infty$, $\mathbb{R}_+ = [0, \infty)$.

Let the functions $\alpha_i,\ \beta_i : [x_0, X) \to \mathbb{R}_+,\ i = 1, 2, \ldots, n$ be such that $\alpha_i(x) \le x$ and $\beta_i(x) \le x$.

Consider the sets $G,\ \Psi,\ \Lambda$ defined by

$$\begin{aligned} G &= \left\{(x,y) \in \mathbb{R}^2 : x \in [x_0, X),\ y \in [y_0, Y)\right\}, \\ \Psi &= \left\{(x,y) \in \mathbb{R}^2 : x \in [J-h, x_0],\ y \in [y_0, Y)\right\}, \\ \Lambda &= \left\{(x,y) \in \mathbb{R}^2 : x \in [J-h, X),\ y \in [y_0, Y)\right\} = G \cup \Psi, \end{aligned}$$

where $J = \min_{1 \le i \le n} \big(\alpha_i(x_0), \beta_i(x_0)\big)$.

Theorem 1. *Let the following conditions be fulfilled:*

1. The functions $\alpha_i, \beta_i \in C^1([x_0, X), \mathbb{R}_+)$ are nondecreasing and the inequalities $\alpha_i(x) \le x$, $\beta_i(x) \le x$ hold on $[x_0, X)$ for $i = 1, \ \ldots, \ n$.

2. The functions $f_i, g_i \in C([J, X) \times [y_0, Y), \mathbb{R}_+)$ for $i = 1, \ \ldots, \ n$.

3. The function $\phi \in C(\Psi, \mathbb{R}_+)$.

4. The function $k \in C(G, (0, \infty))$ is nondecreasing in its both arguments and the inequality $\max_{s \in [J-h, x_0]} \phi(s, y) \le k(x_0, y)$ holds for $y \in [y_0, Y)$.

5. The function $u \in C(\Lambda, \mathbb{R}_+)$ and satisfies the inequalities

$$\begin{aligned} u(x,y) \le k(x,y) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t) dt\, ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h, s]} u(\xi, t) dt\, ds \qquad \text{for } (x,y) \in G, \end{aligned} \tag{1}$$

$$u(x,y) \le \phi(x,y) \qquad \text{for } (x,y) \in \Psi. \tag{2}$$

Then for $(x,y) \in G$ the inequality

$$u(x,y) \le k(x,y) \exp\Big(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) dt\, ds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) dt\, ds \Big) \tag{3}$$

holds.

Proof. From inequality (1) we obtain

$$\begin{aligned}\frac{u(x,y)}{k(x,y)} \leq 1 &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) \frac{u(s,t)}{k(x,y)}\, dt\, ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \frac{\max_{\xi\in[s-h,\,s]} u(\xi,t)}{k(x,y)}\, dt\, ds, \quad (x,y) \in G.\end{aligned} \tag{4}$$

From the monotonicity of the function $k(x,y)$ we obtain for $(x,y) \in G$, $t \in [y_0, Y)$ and $s \in [J, X)$ the inequality

$$\begin{aligned}\frac{\max_{\xi\in[s-h,\,s]} u(\xi,t)}{k(x,y)} &\leq \frac{\max_{\xi\in[s-h,\,s]} u(\xi,t)}{\tilde{k}(s,y)} \leq \max_{\xi\in[s-h,\,s]} \frac{u(\xi,t)}{\tilde{k}(s,y)} \\ &\leq \max_{\xi\in[s-h,\,s]} \frac{u(\xi,t)}{\tilde{k}(\xi,t)},\end{aligned} \tag{5}$$

where the continuous function $\tilde{k} : \Lambda \to \mathbb{R}_+$ is nondecreasing in its both arguments and it is defined by

$$\tilde{k}(x,y) = \begin{cases} k(x,y) & \text{for } (x,y) \in (x_0, X) \times [y_0, Y), \\ k(x_0,y) & \text{for } (x,y) \in \Psi. \end{cases}$$

Define a function $\varphi : \Lambda \to \mathbb{R}_+$ by the equality $\varphi(x,y) = \frac{u(x,y)}{\tilde{k}(x,y)}$.

From inequalities (4), (5) and the definition of the function $\varphi(x,y)$ it follows that

$$\begin{aligned}\varphi(x,y) \leq 1 &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\varphi(s,t)dt\, ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} \varphi(\xi,t)dt\, ds \quad \text{for } (x,y) \in G,\end{aligned} \tag{6}$$

$$\varphi(x,y) = \frac{u(x,y)}{k(x_0,y)} \leq \frac{\phi(x,y)}{k(x_0,y)} \leq 1 \qquad \text{for } (x,y) \in \Psi. \tag{7}$$

Change the variable $s = \alpha_i(\eta)$ for $i = 1, \ldots, n$ in the first integral of inequality (6) and $s = \beta_i(\eta)$ for $i = 1, \ldots, n$ in the second integral of inequality (6), use the condition 1, and obtain

$$\begin{aligned}\varphi(x,y) \leq 1 &+ \sum_{i=1}^{n} \int_{x_0}^{x} \int_{y_0}^{y} f_i(\alpha_i(\eta),t)\varphi(\alpha_i(\eta),t)(\alpha_i(\eta))'\, dt\, d\eta \\ &+ \sum_{i=1}^{n} \int_{x_0}^{x} \int_{y_0}^{y} g_i(\beta_i(\eta),t)\varphi(\beta_i(\eta),t)(\beta_i(\eta))'\, dt\, d\eta.\end{aligned} \tag{8}$$

For any fixed $y \in [y_0, Y)$ we consider the function

$$\begin{aligned} w(x) &= 1 + \sum_{i=1}^{n} \int_{x_0}^{x} \left\{ \int_{y_0}^{y} f_i(\alpha_i(s), t)(\alpha_i(s))' dt \right\} \varphi(y, s) ds \\ &\quad + \sum_{i=1}^{n} \int_{x_0}^{x} \left\{ \int_{y_0}^{y} g_i(\beta_i(s), t)(\beta_i(s))' dt \right\} \varphi(y, s) ds \\ &\leq 1 + \sum_{i=1}^{n} \int_{x_0}^{x} \left\{ \int_{y_0}^{y} f_i(\alpha_i(s), t)(\alpha_i(s))' dt \right\} w(s) ds \\ &\quad + \sum_{i=1}^{n} \int_{x_0}^{x} \left\{ \int_{y_0}^{y} g_i(\beta_i(s), t)(\beta_i(s))' dt \right\} w(s) ds. \end{aligned} \tag{9}$$

From (9) according to Gronwall inequality we have

$$w(x) \leq \exp\left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s, t) dt\, ds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s, t) dt\, ds \right). \tag{10}$$

From inequalities (10), $\varphi(x, y) \leq w(x)$ for any fixed $y \in [y_0, Y)$ and the definitions of functions $\tilde{k}(x, y)$ and $\varphi(x, y)$ we get the required inequality (3). □

Remark 1. Note that the inequality (3) in Theorem 1 is true if the function $k \in C(G, (0, \infty))$ in condition 4 of Theorem 1 is replaced by a function $k \in C(G, \mathbb{R}_+)$.

Corollary 1. *Let the following conditions be fulfilled:*

1. The function $\alpha \in C^1([x_0, X), \mathbb{R}_+)$ is nondecreasing and the inequality $\alpha(x) \leq x$ holds on $[x_0, X)$.

2. The functions $f \in C([x_0, X) \times [y_0, Y), \mathbb{R}_+)$, $g \in C([\alpha(x_0), X) \times [y_0, Y), \mathbb{R}_+)$.

3. The function $\phi \in C([\alpha(x_0) - h, x_0] \times [y_0, Y), \mathbb{R}_+)$.

4. The function $k \in C(G, (0, \infty))$ is nondecreasing in its both arguments and the inequality $\max_{s \in [\alpha(x_0) - h,\, x_0]} \phi(s, y) \leq k(x_0, y)$ holds for $y \in [y_0, Y)$.

5. The function $u \in C([\alpha(x_0) - h, X) \times [y_0, Y), \mathbb{R}_+)$ and satisfies the inequalities

$$\begin{aligned} u(x, y) &\leq k(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t) u(s, t) dt\, ds \\ &\quad + \int_{\alpha(x_0)}^{\alpha(x)} \int_{y_0}^{y} g(s, t) \max_{\xi \in [s-h,\, s]} u(\xi, t) dt\, ds \qquad \text{for } (x, y) \in G, \end{aligned} \tag{11}$$

$$u(x, y) \leq \phi(x, y) \qquad \text{for } (x, y) \in \Psi. \tag{12}$$

Then for $(x, y) \in G$ the inequality

$$u(x, y) \leq k(x, y) \exp\left(\int_{x_0}^{x} \int_{y_0}^{y} f(s, t) dt\, ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{y_0}^{y} g(s, t) dt\, ds \right) \tag{13}$$

holds.

In the case when the unknown function is represented on a power, we obtain the following result.

Theorem 2. *Let the following conditions be fulfilled:*

1. The conditions 1, 2 and 3 of Theorem 1 are satisfied.

2. The function $k \in C(G, (0, \infty))$ *is nondecreasing in its both arguments and the inequality* $\max_{s \in [J-h, x_0]} \phi(s, y) \leq \sqrt[p]{k(x_0, y)}$ *holds for* $y \in [y_0, Y)$.

3. The function $u \in C(\Lambda, \mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u^p(x,y) \leq k(x,y) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t) dt\, ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} u(\xi, t) dt\, ds \quad \text{for } (x,y) \in G, \end{aligned} \tag{14}$$

$$u(x,y) \leq \phi(x,y) \qquad \text{for } (x,y) \in \Psi, \tag{15}$$

where p =*const*≥ 1.

Then for $(x,y) \in G$ *the inequality*

$$u(x,y) \leq \sqrt[p]{k(x,y)} \Big(1 + A(x,y)\Big) \exp\Big(A(x,y)\Big) \tag{16}$$

holds, where

$$A(x,y) = \frac{1}{p} \left(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} \frac{f_i(s,t)}{\big(K(s,t)\big)^{1-\frac{1}{p}}} dtds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} \frac{g_i(s,t)}{\big(K(s,t)\big)^{1-\frac{1}{p}}} dtds \right), \tag{17}$$

$$K(x,y) = \begin{cases} k(x,y), & (x,y) \in (x_0, X) \times [y_0, Y), \\ k(x_0,y), & (x,y) \in [J, x_0] \times [y_0, Y). \end{cases} \tag{18}$$

Proof. Define a function $z: \ \Lambda \to \mathbb{R}_+$ by the equalities

$$z(x,y) = \begin{cases} \frac{\sqrt[p]{k(x,y)}}{p\ k(x,y)} \Big(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t) dt\, ds \\ \quad + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} u(\xi,t) dt\, ds \Big) \\ \qquad\qquad\qquad\qquad\qquad \text{for } (x,y) \in G, \\ 0 \qquad\qquad\qquad\qquad\qquad \text{for } (x,y) \in \Psi. \end{cases} \tag{19}$$

From inequality (14) we obtain for $(x,y) \in G$

$$u^p(x,y) \leq k(x,y) \left(1 + p \frac{z(x,y)}{\sqrt[p]{k(x,y)}}\right)$$

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or

$$u(x,y) \leq \sqrt[p]{k(x,y)}\left(1 + p\frac{z(x,y)}{\sqrt[p]{k(x,y)}}\right)^{\frac{1}{p}}.$$

Apply Bernoulli's inequality $(1+x)^a \leq 1 + ax$, where $0 \leq a \leq 1$ and $-1 < x$, and observe that

$$\begin{aligned} u(x,y) &\leq \sqrt[p]{k(x,y)}\left(1 + \frac{z(x,y)}{\sqrt[p]{k(x,y)}}\right) \\ &= \sqrt[p]{k(x,y)} + z(x,y) \\ &= \mu(x,y) + z(x,y), \qquad (x,y) \in G, \end{aligned} \tag{20}$$

and

$$\begin{aligned} u(x,y) &\leq \phi(x,y) \\ &\leq \phi(x,y) + z(x,y) \\ &\leq \mu(x,y) + z(x,y), \qquad (x,y) \in \Psi, \end{aligned} \tag{21}$$

where

$$\mu(x,y) = \begin{cases} \sqrt[p]{k(x,y)}, & (x,y) \in (x_0, X) \times [y_0, Y) \\ \sqrt[p]{k(x_0,y)}, & (x,y) \in \Psi. \end{cases}$$

Since the function $\mu(x,y)$ is increasing, we have for $s \in [J, X)$ and $y \in [y_0, Y)$

$$\max_{\xi \in [s-h,\, s]} u(\xi, y) \leq \mu(s,y) + \max_{\xi \in [s-h,\, s]} z(\xi, y). \tag{22}$$

Let $x \in [x_0, X)$ and $y \in [y_0, Y)$. Then from inequalities (22), $\alpha_i(x) \leq x$, $\beta_i(x) \leq x$ for $i = 1\ , \dots,\ n$, and the monotonicity of the function $\mu(x,y)$ we get

$$\begin{aligned} &\int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} u(\xi,t)\,dt\,ds \\ &\leq \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t)\left(\mu(s,t) + \max_{\xi \in [s-h,\, s]} z(\xi,t)\right)dt\,ds \\ &\leq \mu(x,y) \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t)\,dt\,ds + \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} z(\xi,t)\,dt\,ds \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t)\,dt\,ds \\ &\leq \mu(x,y) \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) z(s,t)\,dt\,ds. \end{aligned} \tag{24}$$

From the definition of the function $z(x,y)$ and inequalities (23), (24) it follows that

$$\begin{aligned} z(x,y) \le\ & \mu(x,y)A(x,y) + \frac{1}{p}\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y}\frac{f_i(s,t)}{\big(K(s,t)\big)^{1-\frac{1}{p}}}z(s,t)dt\,ds \\ & + \frac{1}{p}\sum_{i=1}^{n}\int_{\beta_i(x_0)}^{\beta_i(x)}\int_{y_0}^{y}\frac{g_i(s,t)}{\big(K(s,t)\big)^{1-\frac{1}{p}}}\max_{\xi\in[s-h,s]}z(\xi,t)dt\,ds, \quad (x,y)\in G, \end{aligned} \tag{25}$$

$$z(x,y) \le 0, \qquad (x,y)\in\Psi, \tag{26}$$

where $A(x,y)$ and $K(x,y)$ are defined by (17) and (18), respectively.

The function $\mu(x,y)A(x,y) \in C(G,\mathbb{R}_+)$ is nondecreasing in its both arguments. Then from inequalities (25) and (26) according to Theorem 1 we get

$$z(x,y) \le \mu(x,y)A(x,y)\exp\Big(A(x,y)\Big) \quad \text{for} \quad (x,y)\in G. \tag{27}$$

Substituting the bound (27) for $z(x,y)$ into the right part of (20), we obtain the required inequality (16). □

Remark 2. Note that in the case $p=1$ the bound (3) in Theorem 1 is better than the bound (16), obtained in Theorem 2.

Corollary 2. *Let the following conditions be fulfilled:*

1. The conditions 1, 2 and 3 of the Corollary 1 are satisfied.

2. The function $k \in C(G,(0,\infty))$ is nondecreasing in its both arguments and the inequality $\max_{s\in[\alpha(x_0)-h,\,x_0]}\phi(s,y) \le \sqrt[p]{k(x_0,y)}$ holds for $y\in[y_0,Y)$.

3. The function $u \in C([\alpha(x_0)-h,X)\times[y_0,Y),\mathbb{R}_+)$ and satisfies the inequalities

$$\begin{aligned} u^p(x,y) \le k(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y}f(s,t)u(s,t)dt\,ds \\ &+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y}g(s,t)\max_{\xi\in[s-h,\,s]}u(\xi,t)dt\,ds \qquad \text{for } (x,y)\in G, \end{aligned} \tag{28}$$

$$u(x,y) \le \phi(x,y) \qquad \text{for } (x,y)\in[\alpha(x_0)-h,x_0]\times[y_0,Y), \tag{29}$$

where $p=$const≥ 1.

Then for $(x,y)\in G$ the inequality

$$u(x,y) \le \sqrt[p]{k(x,y)}\Big(1+B(x,y)\Big)\exp\Big(B(x,y)\Big) \tag{30}$$

holds, where $K(x,y)$ is defined by (18) and

$$B(x,y) = \frac{1}{p}\bigg(\int_{x_0}^{x}\int_{y_0}^{y}\frac{f(s,t)}{\big(k(s,t)\big)^{1-\frac{1}{p}}}dt\,ds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{y_0}^{y}\frac{g(s,t)}{\big(K(s,t)\big)^{1-\frac{1}{p}}}dt\,ds\bigg).$$

In the case of a non-monotonic function $k(x,y)$ we obtain the following result.

Theorem 3. *Let the following conditions be fulfilled:*
1. The conditions 1 and 2 of Theorem 1 are satisfied.
2. The function $q \in C(G, \mathbb{R}_+)$.
3. The function $k \in C(\Lambda, \mathbb{R}_+)$.
4. The function $u \in C(\Lambda, \mathbb{R}_+)$ and satisfies the inequalities

$$\begin{aligned} u(x,y) \le k(x,y) + q(x,y)\Big(&\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t)dt\,ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} u(\xi,t)dt\,ds \Big) \quad \text{for } (x,y)\in G, \end{aligned} \tag{31}$$

$$u(x,y) \le k(x,y) \quad \text{for } (x,y)\in \Psi. \tag{32}$$

Then for $(x,y)\in G$ the inequality

$$u(x,y) \le k(x,y) + e(x,y)q(x,y)\exp\Big(E(x,y)\Big) \tag{33}$$

holds, where

$$\begin{aligned} e(x,y) = &\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) k(s,t)dt\,ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} k(\xi,t)dt\,ds \quad \text{for } (x,y)\in G, \end{aligned} \tag{34}$$

$$\begin{aligned} E(x,y) = &\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) Q(s,t)dt\,ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} Q(\xi,t)dt\,ds \quad \text{for } (x,y)\in G, \end{aligned} \tag{35}$$

$$Q(x,y) = \begin{cases} q(x,y) & \text{for } (x,y)\in G, \\ q(x_0,y) & \text{for } (x,y)\in \Psi. \end{cases} \tag{36}$$

Proof. Define a function $z: \ \Lambda \to \mathbb{R}_+$ by the equalities

$$z(x,y) = \begin{cases} \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t)dt\,ds \\ \quad + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} u(\xi,t)dt\,ds \\ \qquad\qquad\qquad\qquad \text{for } (x,y)\in G \\ 0 \qquad\qquad\qquad\qquad \text{for } (x,y)\in \Psi. \end{cases} \tag{37}$$

From inequality (31) and the definition of function $z(x, y)$ we have

$$u(x, y) \leq k(x, y) + Q(x, y)z(x, y) \text{ for } (x, y) \in \Lambda. \tag{38}$$

Let $x \in [x_0, X)$ and $y \in [y_0, Y)$. Then from inequalities (38), $\alpha_i(x) \leq x$, $\beta_i(x) \leq x$ for $i = 1, \dots, n$ we get

$$\begin{aligned} &\int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} u(\xi, t) dt\, ds \\ &\leq \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} k(\xi, t) dt\, ds \\ &\quad + \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} Q(\xi, t) \max_{\xi \in [s-h,\, s]} z(\xi, t) dt\, ds. \end{aligned} \tag{39}$$

and

$$\begin{aligned} &\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) u(s,t) dt\, ds \\ &\leq \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) k(s,t) dt\, ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) Q(s,t) z(s,t) dt\, ds. \end{aligned} \tag{40}$$

From the definition of the function $z(x, y)$ and inequalities (39), (40) it follows that

$$\begin{aligned} z(x,y) \leq e(x,y) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t) Q(s,t) z(s,t) dt\, ds \\ &+ \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi \in [s-h,\, s]} Q(\xi, t) \max_{\xi \in [s-h,\, s]} z(\xi, t) dt\, ds \\ &\qquad \text{for } (x, y) \in G, \end{aligned} \tag{41}$$

$$z(x, y) = 0 \qquad \text{for } (x, y) \in \Psi, \tag{42}$$

where the function $e(x, y)$ is defined by equality (34). Note the function $e \in C(G, \mathbb{R}_+)$ is nondecreasing in its both arguments and $e(x_0, y) \equiv 0$ for $y \in [y_0, Y)$.

From inequalities (41) and (42) according to Theorem 1 we get

$$z(x, y) \leq e(x, y) \exp\left(E(x, y)\right), \tag{43}$$

where the function $E(x, y)$ is defined by (35).

Inequalities (38) and (43) imply the validity of the required inequality (33). □

Corollary 3. *Let the following conditions be fulfilled:*

1. The conditions 1 and 2 of Theorem 1 are satisfied.

2. The function $k \in C(\Lambda, \mathbb{R}_+)$ is nondecreasing.

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3. The function $u \in C(\Lambda, \mathbb{R}_+)$ and satisfies the inequality

$$u(x,y) \le q(x,y)\Big(k(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)u(s,t)dt\,ds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} u(\xi,t)dt\,ds\Big) \quad \text{for } (x,y)\in G, \tag{44}$$

and

$$u(x,y) \le q(x,y)k(x,y) \quad \text{for } (x,y)\in\Psi. \tag{45}$$

Then for $(x,y)\in G$ the inequality

$$u(x,y) \le q(x,y)k(x,y)\Big(1+E(x,y)\Big)\exp\Big(E(x,y)\Big) \tag{46}$$

holds, where $E(x,y)$ is defined by (35).

In the case when the unknown function is represented on a power, and the function $k(x,y)$ is not monotonic, we obtain the following result which proof is similar to the proof of Theorem 2 and it is based on the result of Corollary 3.

Theorem 4. *Let the following conditions be fulfilled:*
1. The conditions 1 and 2 of Theorem 1 are satisfied.
2. The function $k \in C(\Lambda, (0,\infty))$.
3. The function $u \in C(\Lambda, \mathbb{R}_+)$ and satisfies the inequalities

$$u^p(x,y) \le k(x,y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)u(s,t)dt\,ds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,\,s]} u(\xi,t)dt\,ds \quad \text{for } (x,y)\in G, \tag{47}$$

$$u(x,y) \le \sqrt[p]{k(x,y)} \quad \text{for } (x,y)\in\Psi, \tag{48}$$

where p =const≥ 1.

Then for $(x,y)\in G$ the inequality

$$u(x,y) \le \sqrt[p]{k(x,y)}\Big(1+\frac{1}{p}\,\frac{M(x,y)}{k(x,y)}\Big(1+S(x,y)\Big)\exp\Big(S(x,y)\Big)\Big) \tag{49}$$

holds, where

$$M(x,y) = \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\sqrt[p]{k(s,t)}dt\,ds + \sum_{i=1}^{n} \int_{\beta_i(x_0)}^{\beta_i(x)} \int_{y_0}^{y} g_i(s,t) \max_{\xi\in[s-h,s]} \sqrt[p]{k(\xi,t)}dt\,ds, \tag{50}$$

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$$S(x,y) = \frac{1}{p}\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y}\frac{f_i(s,t)}{\Big(k(s,t)\Big)^{1-\frac{1}{p}}}\,dt\,ds + \sum_{i=1}^{n}\int_{\beta_i(x_0)}^{\beta_i(x)}\int_{y_0}^{y}\frac{g_i(s,t)}{\Big(\max_{\xi\in[s-h,\,s]}k(\xi,t)\Big)^{1-\frac{1}{p}}}\,dt\,ds. \tag{51}$$

Proof. Define a function $z:\ \Lambda\to\mathbb{R}_+$ by the equalities (37).

From inequality (47) and Bernoulli's inequality we obtain for $(x,y)\in\Lambda$

$$u(x,y)\leq \sqrt[p]{k(x,y)} + z(x,y). \tag{52}$$

We have for $s\in[J,X)$ and $y\in[y_0,Y)$

$$\max_{\xi\in[s-h,\,s]} u(\xi,y)\leq \max_{\xi\in[s-h,\,s]}\sqrt[p]{k(\xi,y)} + \max_{\xi\in[s-h,\,s]} z(\xi,y). \tag{53}$$

Let $x\in[x_0,X)$ and $y\in[y_0,Y)$. Then from inequalities (22), $\alpha_i(x)\leq x$, $\beta_i(x)\leq x$ for $i=1,\ \ldots,\ n$, and the monotonicity of the function $k(x,y)$ we get

$$\begin{aligned}\int_{\beta_i(x_0)}^{\beta_i(x)}\int_{y_0}^{y} g_i(s,t)\max_{\xi\in[s-h,\,s]} u(\xi,t)dt\,ds &\leq \int_{\beta_i(x_0)}^{\beta_i(x)}\int_{y_0}^{y} g_i(s,t)\max_{\xi\in[s-h,\,s]}\sqrt[p]{k(\xi,y)}dt\,ds\\ &\quad+\int_{\beta_i(x_0)}^{\beta_i(x)}\int_{y_0}^{y} g_i(s,t)\max_{\xi\in[s-h,\,s]} z(\xi,t)dt\,ds\end{aligned} \tag{54}$$

and

$$\begin{aligned}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)u(s,t)dt\,ds &\leq \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)\sqrt[p]{k(s,y)}dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)z(s,t)dt\,ds.\end{aligned} \tag{55}$$

From the definition of the function $z(x,y)$ and inequalities (54), (55) it follows that

$$\begin{aligned}z(x,y)\leq\ &\frac{1}{p\ (k(x,y))^{1-\frac{1}{p}}}\Big(M(x,y)+\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{y_0}^{y} f_i(s,t)z(s,t)dt\,ds\\ &+\sum_{i=1}^{n}\int_{\beta_i(x_0)}^{\beta_i(x)}\int_{y_0}^{y} g_i(s,t)\max_{\xi\in[s-h,\,s]} z(\xi,t)dt\,ds\Big)\quad \text{for } (x,y)\in G,\end{aligned} \tag{56}$$

$$z(x,y)=0 \qquad\qquad \text{for } (x,y)\in\Psi, \tag{57}$$

where $M(x,y)$ is defined by (50).

From inequalities (56) and (57) according to Corollary 3 we get for $(x,y) \in G$

$$z(x,y) \leq \frac{1}{p\ (k(x,y))^{1-\frac{1}{p}}} M(x,y)\Big(1 + S(x,y)\Big) \exp\Big(S(x,y)\Big), \tag{58}$$

where $S(x,y)$ is defined by (51).

Substituting the bound (58) for $z(x,y)$ into the right hand part of (52), we obtain the required inequality (49).

Remark 3. Note in the case of nondecreasing function $k(x,y)$ the bound (16) in Theorem 2 is better than the bound (49) in Theorem 4.

3. Application

We will apply some of the solved above inequalities to obtain bounds for the solutions of partial differential equations with "maxima".

Let both functions $\sigma,\ \tau \in C([x_0,X),\mathbb{R}_+)$ be given such that $\tau(x)$ is an increasing function, $\sigma(x) \leq \tau(x) \leq x$ and there exists a constant $h > 0 :\ \tau(x) - \sigma(x) \leq h$ for $x \geq x_0$.

Example 1. Consider the following scalar partial differential equation with "maxima"

$$u''_{x\,y} = F\Big(x,\ y,\ u(x,y), \max_{s\in[\sigma(x),\,\tau(x)]} u(s,y)\Big) \quad \text{for} \quad x \geq x_0, \quad y \geq y_0, \tag{59}$$

with the initial conditions

$$\begin{aligned} u(x_0,y) &= \varphi_1(y) && \text{for } y \in [y_0,Y), \\ u(x,y_0) &= \varphi_2(x) && \text{for } x \in [x_0,X), \\ u(x,y) &= \psi(x,y) && \text{for } (x,y) \in [\tau(x_0)-h, x_0] \times [y_0,Y), \end{aligned} \tag{60}$$

where $u \in \mathbb{R}$, $\varphi_1 : [y_0,Y) \to \mathbb{R}$, $\varphi_2 : [x_0,X) \to \mathbb{R}$, $\psi : [\tau(x_0)-h,x_0] \times [y_0,Y) \to \mathbb{R}$, $F : [x_0,X) \times [y_0,Y) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Theorem 5. (Uniqueness) *Let the following conditions be fulfilled:*

1. The functions $\sigma,\ \tau \in C([x_0,X),\mathbb{R}_+)$, $\tau(x)$ *is an increasing function,* $\sigma(x) \leq \tau(x) \leq x$ *and there exists a constant* $h > 0 :\ 0 < \tau(x) - \sigma(x) \leq h$ *for* $x \geq x_0$.

2. The function $F \in C([x_0,X) \times [y_0,Y) \times \mathbb{R} \times \mathbb{R},\mathbb{R})$ *and satisfies for* $x \geq x_0$, $y \geq y_0$ *and* $\gamma_i,\ \zeta_i \in \mathbb{R},\ (i = 1,2)$ *the condition*

$$\big|F(x,\ y,\ \gamma_1,\ \zeta_1) - F(x,\ y,\ \gamma_2,\ \zeta_2)\big| \leq f(x,y)\big|\gamma_1 - \gamma_2\big| + g(x,y)\big|\zeta_1 - \zeta_2\big|,$$

where $f,\ g \in C([x_0,X) \times [y_0,Y),\mathbb{R}_+)$.

3. The function $\psi \in C([\tau(x_0)-h, x_0] \times [y_0, Y), \mathbb{R})$.

4. The functions $\varphi_1 \in C^1([y_0, Y), \mathbb{R})$, $\varphi_2 \in C^1([x_0, X), \mathbb{R})$ *and the equalities* $\varphi_1(y_0) = \varphi_2(x_0)$, $\varphi_1(y) = \psi(x_0, y)$, $y \in [y_0, Y)$ *hold.*

5. The initial value problem (59), (60) has at least one solution, defined for $(x, y) \in [\tau(x_0)-h, X) \times [y_0, Y)$.

Then the partial differential equation with "maxima" (59), (60) has a unique solution.

Proof. Assume that there exist two different solutions $v(x, y)$ and $w(x, y)$ of (59), (60) which are defined for $(x, y) \in [\tau(x_0)-h, X) \times [y_0, Y)$. Both functions satisfy the following integral equation

$$\begin{aligned} u(x,y) &= \varphi_1(y) + \varphi_2(x) - \varphi_2(x_0) \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} F\big(s,\ t,\ u(s,t), \max_{\xi\in[\sigma(s),\,\tau(s)]} u(\xi,t)\big)dt\,ds \quad \text{for } (x,y)\in G, \\ u(x,y) &= \psi(x,y) \qquad \text{for } (x,y) \in [\tau(x_0)-h, x_0]\times[y_0,Y). \end{aligned} \tag{61}$$

Consider the difference of both solutions $U(x, y) = \big|v(x, y) - w(x, y)\big|$ and using condition 2 of Theorem 5, we obtain

$$\begin{aligned} U(x,y) &\le \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)U(s,t)dt\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\Big|\max_{\xi\in[\sigma(s),\,\tau(s)]} v(\xi,t) - \max_{\xi\in[\sigma(s),\,\tau(s)]} w(\xi,t)\Big|dt\,ds \\ &\le \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)U(s,t)dt\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\max_{\xi\in[\sigma(s),\tau(s)]} U(\xi,t)dt\,ds \qquad \text{for } (x,y)\in G, \end{aligned} \tag{62}$$

$$U(x,y) = 0 \qquad \text{for } (x,y) \in [\tau(x_0)-h, x_0]\times[y_0,Y). \tag{63}$$

Change the variable $s = \tau^{-1}(\eta)$ in the second integral of (62), use the inequality $\max_{\xi\in[\sigma(s),\,\tau(s)]} U(\xi, y) \le \max_{\xi\in[\tau(s)-h,\,\tau(s)]} U(\xi, y)$ for $y \in [y_0, Y)$ and $s \in [x_0, X)$ that follows from condition 1 of Theorem 5, and obtain for $(x, y) \in G$ the following inequality

$$\begin{aligned} U(x,y) \le{}& \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)U(s,t)dt\,ds \\ &+ \int_{\tau(x_0)}^{\tau(x)}\int_{y_0}^{y} g(\tau^{-1}(\eta),t)\big(\tau^{-1}(\eta)\big)' \max_{\xi\in[\eta-h,\,\eta]} U(\xi,t)dt\,d\eta. \end{aligned} \tag{64}$$

From inequalities (63), (64) according to Corollary 1 for $k(x, y) \equiv 0$ we obtain $U(x, y) \equiv 0$. □

Theorem 6. (Upper Bound) *Let the following conditions be fulfilled:*

1. The conditions 1, 3, 4 and 5 of Theorem 5 are satisfied.

2. The function $F \in C([x_0, X) \times [y_0, Y) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and satisfies for* $x \geq x_0$, $y \geq y_0$ *and* $\gamma,\ \zeta \in \mathbb{R}$, *the condition* $\big|F(x,\ y,\ \gamma,\ \zeta)\big| \leq f(x,y)\big|\gamma\big| + g(x,y)\big|\zeta\big|$, *where* $f,\ g \in C([x_0, X) \times [y_0, Y), \mathbb{R}_+)$.

Then for $(x,y) \in G$ *the solution of the initial value problem (59), (60) for the differential equation with "maxima" satisfies the inequality*

$$\big|u(x,y)\big| \leq \tilde{K}(x,y) + e(x,y)\exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\Big(f(s,t)+g(s,t)\Big)dt\,ds\right) \tag{65}$$

holds, where

$$e(x,y) = \int_{x_0}^{x}\int_{y_0}^{y}\Big(f(s,t)\tilde{K}(s,t) + g(s,t)\max_{\xi\in[s-h,\,s]}\tilde{K}(\xi,t)\Big)dt\,ds,$$

$$\tilde{K}(x,y) = \begin{cases} |\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)| & \text{for } (x,y)\in G,\\ \psi(x,y) & \text{for } (x,y)\in[\tau(x_0)-h, x_0]\times[y_0,Y). \end{cases}$$

Proof. From the integral equation (61) and the condition 2 of Theorem 6 we obtain

$$\begin{aligned} |u(x,y)| &= |\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)| + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)|u(s,t)|dt\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)|\max_{\xi\in[\sigma(s),\,\tau(s)]} u(\xi,t)|dt\,ds \qquad \text{for } (x,y)\in G, \\ |u(x,y)| &= |\psi(x,y)| \qquad\qquad \text{for } (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y), \end{aligned}$$

or we get for $(x,y) \in G$

$$\begin{aligned} |u(x,y)| \leq{}& |\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)| + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)|u(s,t)|dt\,ds \\ &+ \int_{\tau(x_0)}^{\tau(x)}\int_{y_0}^{y} g(\tau^{-1}(\eta),t)(\tau^{-1}(\eta))'\max_{\xi\in[\eta-h,\,\eta]}|u(\xi,t)|dt\,d\eta. \end{aligned} \tag{66}$$

From inequality (66) according to Theorem 3 we obtain the bound (65).

Remark 4. If in Theorem 6 the functions $\varphi_1 \in C^1([y_0, Y), \mathbb{R})$, $\varphi_2 \in C^1([x_0, X), \mathbb{R})$, $\psi(x,y) \in C([\tau(x_0)-h, x_0]\times[y_0,Y),\mathbb{R})$ are nondecreasing, then applying Corollary 1 instead of Theorem 3 we obtain a better bound for the solution of the initial value problem (59), (60) for the differential equation with "maxima" :

$$\big|u(x,y)\big| \leq |\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)|\exp\left(\int_{x_0}^{x}\int_{y_0}^{y}\Big(f(s,t)+g(s,t)\Big)dt\,ds\right).$$

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Example 2. Consider the following scalar partial differential equation with "maxima"

$$uu_{xy} + u_x u_y = F\Big(x,\ y,\ u(x,y), \max_{s\in[\sigma(x),\,\tau(x)]} u(s,y)\Big) \quad \text{for} \quad x \geq x_0, \quad y \geq y_0 \tag{67}$$

with the initial conditions (60), where $u \in \mathbb{R}$, $F : [x_0, X) \times [y_0, Y) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Theorem 7. (Upper Bound) *Let the following conditions be fulfilled:*

1. The conditions 1, 3, 4 and 5 of Theorem 5 are satisfied.

2. The function $F \in C([x_0, X) \times [y_0, Y) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and satisfies for $x \geq x_0$, $y \geq y_0$ and $\gamma,\ \zeta \in \mathbb{R}$, the condition $\big|F(x,\ y,\ \gamma,\ \zeta)\big| \leq f(x,y)\big|\gamma\big| + g(x,y)\big|\zeta\big|$, where $f,\ g \in C([x_0, X) \times [y_0, Y), \mathbb{R}_+)$.

Then for $(x,y) \in G$ the solution of the initial value problem (67), (60) for the differential equation with "maxima" satisfies the inequality

$$\begin{aligned} |u(x,y)| \leq &\sqrt{\big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2} \\ &+ \frac{M(x,y)}{\sqrt{\big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2}}\Big(1 + S(x,y)\Big)e^{S(x,y)} \end{aligned} \tag{68}$$

holds, where

$$\begin{aligned} M(x,y) = &\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\sqrt{\big(\varphi_1(t)\big)^2 + \big(\varphi_2(s)\big)^2 - \big(\varphi_2(x_0)\big)^2}\,dtds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\sqrt{\big(\varphi_1(t)\big)^2 + \max_{\xi\in[s-h,s]}\big(\varphi_2(\xi)\big)^2 - \big(\varphi_2(x_0)\big)^2}\,dtds, \\ S(x,y) = &\int_{x_0}^{x}\int_{y_0}^{y} \frac{f(s,t)}{\sqrt{\big(\varphi_1(t)\big)^2 + \big(\varphi_2(s)\big)^2 - \big(\varphi_2(x_0)\big)^2}}\,dt\,ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} \frac{g(s,t)}{\sqrt{\big(\varphi_1(t)\big)^2 + \max_{\xi\in[s-h,\,s]}\big(\varphi_2(\xi)\big)^2 - \big(\varphi_2(x_0)\big)^2}}\,dt\,ds. \end{aligned} \tag{69}$$

Proof. The solution $u(x,y)$ of the initial value problem (67), (60) satisfies the following integral equation

$$\begin{aligned} \big(u(x,y)\big)^2 &= \big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2 \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} 2F\Big(s,\ t,\ u(s,t), \max_{\xi\in[\sigma(s),\,\tau(s)]} u(\xi,t)\Big)dt\,ds \\ &\qquad\qquad \text{for } (x,y) \in G, \\ u(x,y) &= \psi(x,y) \qquad \text{for } (x,y) \in [\tau(x_0) - h, x_0] \times [y_0, Y). \end{aligned}$$

According to condition 2 of Theorem 7 we get for $(x,y) \in G$

$$\begin{aligned}\big(u(x,y)\big)^2 \le \big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2 \\ + 2\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\big|u(s,t)\big|dt\,ds \\ + 2\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\max_{\xi\in[\sigma(s),\,\tau(s)]}\big|u(\xi,t)\big|dt\,ds.\end{aligned} \tag{70}$$

Set $U(x,y) = \big|u(x,y)\big|$ for $(x,y) \in [\tau(x_0)-h, X) \times [y_0, Y)$, change the variable $s = \tau^{-1}(\eta)$ in the second integral of (70), use the inequality $\max_{\xi\in[\sigma(s),\,\tau(s)]} U(\xi,y) \le \max_{\xi\in[\tau(s)-h,\,\tau(s)]} U(\xi,y)$ for $y \in [y_0, Y)$ and $s \in [x_0, X)$ that follows from condition 1 of Theorem 5, and obtain for $(x,y) \in G$ the following inequality

$$\begin{aligned}\Big(U(x,y)\Big)^2 \le \big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2 \\ + \int_{x_0}^{x}\int_{y_0}^{y} 2f(s,t)U(s,t)dt\,ds \\ + \int_{\tau(x_0)}^{\tau(x)}\int_{y_0}^{y} 2g(\tau^{-1}(\eta),t)(\tau^{-1}(\eta))'\max_{\xi\in[\eta-h,\,\eta]} U(\xi,t)dt\,d\eta.\end{aligned} \tag{71}$$

According to Theorem 4 from inequality (71) for $p = 2$, $i = 1$, $\alpha(x) \equiv x$ and $\beta(x) \equiv x$ for $x \in [x_0, X)$, $k(x,y) = \big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2 \ge 0$ we obtain the inequality (68). □

Remark 5. If in Theorem 7 the functions $\varphi_1 \in C^1([y_0,Y),\mathbb{R})$, $\varphi_2 \in C^1([x_0,X),\mathbb{R})$, $\psi(x,y) \in C([\tau(x_0)-h, x_0]\times[y_0,Y),\mathbb{R})$ are nondecreasing, then applying Theorem 2 instead of Theorem 4 we obtain better bound for the solution of the initial value problem (67), (60) for the differential equation with "maxima":

$$\big|u(x,y)\big| \le \sqrt{\big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2}\Big(1 + P(x,y)\Big)e^{P(x,y)},$$

where

$$P(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} \frac{f(s,t)+g(s,t)}{\sqrt{\big(\varphi_1(y)\big)^2 + \big(\varphi_2(x)\big)^2 - \big(\varphi_2(x_0)\big)^2}}\,dt\,ds.$$

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