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A FOURTH ORDER SINGULAR THREE POINT BOUNDARY VALUE PROBLEM

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Abstract: The existence of a positive solution is obtained for the fourth order three point boundary value problem, $y^{(4)} + f(x, y) = 0,\ 0 < x \leq 1,\ y(0) = y'(p) = y''(p) = y'''(1) = 0$, where $0 < p < 1$ is fixed and where $f(x, r)$ is singular at $x = 0$, $r = 0$, and possibly at $r = \infty$. The method applies a fixed point theorem for mappings that are decreasing with respect to a cone.

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1. Introduction

In this paper, we consider positive solutions for a boundary value problem for the fourth order ordinary differential equation,

$$y^{(4)} + f(x, y) = 0, \quad 0 < x \leq 1, \tag{1}$$

satisfying the three point boundary conditions,

$$y(0) = y'(p) = y''(p) = y'''(1) = 0, \tag{2}$$

where $0 < p < 1$ is fixed and where $f(x, r)$ is singular at $x = 0, r = 0$, and may be singular at $r = \infty$.

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We assume the following hold for f:

(i) $f(x,r) : (0,1] \times (0,\infty) \to (0,\infty)$ is continuous and decreasing in r for every $x \in (0,1]$,

(ii) $\lim_{r\to 0^+} f(x,r) = \infty$ and $\lim_{r\to\infty} f(x,r) = 0$, uniformly on compact subsets of $(0,1]$.

The study of singular boundary value problems for ordinary differential equations enjoys substantial history since the paper by Gatika, Oliker, and Waltman [3]. They studied singularities of the type in (i)-(ii) for second order Sturm-Louiville boundary value problems. The key for the Gatica, Oliker, Waltman results hinged on an application of a fixed point theorem for operators that are decreasing with respect to a cone. Subsequent works for similar singularities were done by Eloe and Henderson [2], Henderson and Yin [6], [7], Maroun [9], [10], and O'Regan [11] in which right focal, focal, and higher order boundary value problems were considered. In addition, Graef, Henderson and Yang [4], Henderson and Singh [5] and Singh [12] adapted this fixed point theorem for singular nonlocal boundary value problems.

In the present work, the Gatica, Oliker, Waltman fixed point theorem is applied to obtain solutions for (1), (2). Our intent is to transform the boundary value problem into an integral equation by use of an appropriate Green's function, $G(x,t)$, which will play the role of the kernel of operators, T_n, for which we seek fixed points. These fixed points will form a sequence of iterates converging to a solution of the boundary value problem.

In the next section, we give definitions and some properties of cones in a Banach space. We then state the fixed point theorem due to Gatica, Oliker, and Waltman [3].

2. A Fixed Point Theorem

We begin by giving definitions and some properties of cones in a Banach space. For references, see Krasnosel'skii [8] and Amann [1].

Let $\mathcal{B}$ be a real Banach space. A nonempty set $\mathcal{K} \subset \mathcal{B}$ is called a *cone* if the following conditions are satisfied:

(a) the set $\mathcal{K}$ is closed;
(b) if $u, v \in \mathcal{K}$ then $\alpha u + \beta v \in \mathcal{K}$ for all $\alpha, \beta \geq 0$;
(c) $u, -u \in \mathcal{K}$ imply $u = 0$.

Given a cone, $\mathcal{K}$, a *partial order*, $\leq$, is induced on $\mathcal{B}$ by $x \leq y$, for $x,\ y \in \mathcal{B}$ iff $y - x \in \mathcal{K}$ (for clarity, we sometimes write $x \leq y$ (w.r.t. $\mathcal{K}$)). If $x, y \in \mathcal{B}$ with $x \leq y$, let $< x, y >$ denote the *closed order interval between x and y* given by, $< x, y >= \{z \in \mathcal{B} \mid x \leq z \leq y\}$. A cone $\mathcal{K}$ is *normal* in $\mathcal{B}$ provided there exists $\delta > 0$ such that $\| e_1 + e_2 \| \geq \delta$, for all $e_1, e_2 \in \mathcal{K}$ with $\| e_1 \| = \| e_2 \| = 1$.

Remark. If $\mathcal{K}$ is a normal cone in $\mathcal{B}$, then closed order intervals are norm bounded (see Krasnosel'skii, [8]).

We now state the fixed point theorem due to Gatica, Oliker, and Waltman [3].

Theorem 1. *Let $\mathcal{B}$ be a Banach space, $\mathcal{K}$ a normal cone in $\mathcal{B}$, D a subset of $\mathcal{K}$ such that if x, y are elements of D, $x \le y$, then $< x, y >$ is contained in D, and let $T : D \to \mathcal{K}$ be a continuous decreasing mapping which is compact on any closed order interval contained in D. Suppose there exists an $x_0 \in D$ such that $T^2 x_0$ is defined (where $T^2 x_0 = T(Tx_0)$) and furthermore $Tx_0, T^2 x_0$ are (order) comparable to x_0. Then T has a fixed point in D provided that either:*

(I) $Tx_0 \le x_0$ & $T^2 x_0 \le x_0$ *or* $Tx_0 \ge x_0$ & $T^2 x_0 \ge x_0$, *or*

(II) *The complete sequence of iterates $\left\{T^n x_0\right\}_{n=0}^{\infty}$ is defined and there exists $y_0 \in D$ such that $Ty_0 \in D$ and $y_0 \le T^n x_0$, for every n.*

3. The Fourth Order Singular Problem

We look for positive solutions of (1), (2) on $(0, 1]$ of class $C^{(3)}[0, 1] \cap C^{(4)}(0, 1]$. We find fixed points of integral operators associated with a sequence of nonsingular fourth order perturbations of (1), (2). We then show the sequence of fixed points forms a sequence of iterates which converges to a solution of the integral equation associated with (1), (2).

We observe that solutions of (1), (2) are positive and nondecreasing. Somewhat in conjunction with this observation, we now state a couple lemmas due to Yang [13], which will play prominent roles in our future arguments.

Lemma 2. *If $u \in C^{(4)}[0, 1]$ satisfies the boundary conditions* (2) *and is such that*

$$u^{(4)}(x) \le 0, \ \text{for } 0 \le x \le 1, \tag{3}$$

then $u'(x) \ge 0$, for $0 \le x \le 1$, and $0 \le u(x) \le u(1)$, for $0 \le x \le 1$.

Before stating the second Yang lemma [13], we introduce a crucial function $g_1 : [0, 1] \to [0, \infty)$ defined by

$$g_1(x) := \frac{(x - p)^3 + p^3}{3p^2 - 3p + 1}.$$

Lemma 3. *If $u \in C^{(4)}[0, 1]$ safisfies (3) and (2), then*

$$u(x) \ge g_1(x)u(1) \ \text{for } 0 \le x \le 1.$$

We shall consider the Banach space, $\mathcal{B}$, with associated norm,

$$\mathcal{B} = \{u : [0, 1] \to \mathbb{R} \mid u \text{ is continuous}\},$$

$$\| u \| = \sup_{x\in[\,0,\,1\,]} |u(x)|.$$

We define a normal cone, $\mathcal{K}$, in $\mathcal{B}$ as

$$\mathcal{K} := \{\, u \in \mathcal{B} \mid u(x) \geq 0 \text{ on } [0,1]\,\}\,.$$

And finally for $\theta > 0$, let

$$g_\theta(x) := \theta \cdot g_1(x).$$

We observe at this point, for each positive (and nondecreasing) solution, $y(x)$, of (1), (2), for $\theta = y(1) = \|y\| > 0$, it follows from Lemmas 2 and 3 that

$$g_\theta(x) \leq y(x),\ 0 \leq x \leq 1.$$

For the final hypothesis we assume, for all $\theta > 0$,

(iii) $0 < \displaystyle\int_0^1 f(x, g_\theta(x))dx < \infty$.

We will apply Theorem 1 to operators whose kernel is the Green's function for $-y^{(4)} = 0$ and satisfies (2). As shown by Yang [13], this Green's function, $G : [0,1] \times [0,1] \to [0,\infty)$, is given by

$$G(x,t) = \begin{cases} t^3/6, & t \leq p \text{ and } t \leq x\ , \\ ((x-p)^3 + p^3)/6, & t > p \text{ and } t > x\ , \\ ((x-t)^3 + t^3)/6, & t \leq p \text{ and } t > x\ , \\ (p^3 + (t-x)^3 + (x-p)^3)/6, & t > p \text{ and } t \leq x. \end{cases}$$

Notice, if $(x,t) \in (0,1) \times (0,1)$ then $G(x,t) > 0$.

We define a subset, D, of the cone as

$$D := \big\{\phi \in \mathcal{K} \mid \exists\, \theta(\phi) > 0 \text{ such that } \phi(x) \geq g_\theta(x),\ x \in [0,1]\big\}.$$

Moreover, let the integral operator, $T : D \to \mathcal{K}$, be defined by

$$T\phi(x) = \int_0^1 G(x,t) f(t, \phi(t))dt,$$

for $0 \leq x \leq 1$, and $\phi \in D$.

Notice, it suffices to define D, as above, since the singularity in f does not allow definition of our operator T on some of the cone $\mathcal{K}$. Furthermore, it can easily be verified that T is well defined. In that direction, for $\phi \in D$, there exists a $\theta(\phi) > 0$ such that $g_\theta(x) \leq \phi(x),\ \ 0 < x \leq 1$. And since $f(\cdot, r)$ decreases with respect to r, we have

$$\begin{aligned} 0 &\leq \int_0^1 G(x,t) f(t, \phi(t))dt. \\ &\leq \int_0^1 G(x,t) f(t, g_\theta(t))dt \\ &< \infty. \end{aligned}$$

Remark. It is straightforward that T is decreasing with respect to D, and it can be shown by Lemmas 2 and 3 that $\phi \in D$ is a solution of (1), (2) iff $T\phi = \phi$.

We now present a number of lemmas that allow us to apply Theorem 1.

Lemma 4. *If f satisfies (i)-(iii), then there exists an $S > 0$ such that $\| \phi \| \leq S$ for any solution ϕ in D of (1), (2).*

Proof. We prove the lemma by contradiction. Assume the conclusion is false. Then there exists a sequence, $\{\phi_m\}_{m=1}^{\infty}$, of solutions to (1), (2) such that $\phi_m(x) > 0$ for $x \in (0, 1]$, and

$$\| \phi_m \| \leq \| \phi_{m+1} \| \text{ and } \lim_{m\to\infty} \| \phi_m \| = \infty.$$

For $\phi(x)$, a solution of (1), we have $\phi^{(4)}(x) = -f(x, \phi(x))$, $0 < x \leq 1$, for f positive. Moreover, the boundary conditions (2) and Lemma 3 give us

$$\phi_m(x) \geq g_1(x)\phi_m(1) \geq g_1(p)\phi_m(1) = \frac{p^3}{3p^2 - 3p + 1} \| \phi_m \|, \quad p \leq x \leq 1.$$

Now let us define

$$M := \max\{G(x, t) : (x, t) \in [0, 1] \times [0, 1]\}.$$

Then from condition (ii) there exists an m_0 such that if $m \geq m_0$, then

$$f(x, \phi_m(x)) \leq \frac{1}{M(1-p)}, \text{ for } x \in [p, 1].$$

Let $\theta = \phi_{m_0}(1) = \|\phi_{m_0}\|$. Then, for $m \geq m_0$, $\phi_m(x) \geq g_1(x)\|\phi_m\| \geq g_1(x)\|\phi_{m_0}\| = g_\theta(x)$, $0 \leq x \leq 1$. Thus, for $m \geq m_0$ and ϕ_m a solution of (1), (2), we have

$$\begin{aligned}
\phi_m(x) &= T\phi_m(x) \\
&= \int_0^1 G(x, t) f(t, \phi_m(t))dt \\
&\leq \int_0^p G(x, t) f(t, g_\theta(t))dt + \int_p^1 M \frac{1}{M(1-p)} dt \\
&= \int_0^p G(x, t) f(t, g_\theta(t))dt + 1 \\
&< \infty.
\end{aligned}$$

This is a contradiction to the assumption that $\lim_{m\to\infty} \| \phi_m \| = \infty$. Hence, for any solution $\phi \in D$ of (1), (2), there exists an $S > 0$ such that $\| \phi \| \leq S$. □

Lemma 5. *If f satisfies (i)-(iii), then there exists an $R > 0$ such that $\| \phi \| \geq R$ for any solution ϕ in D of (1), (2).*

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

Proof. We assume the conclusion is false. Then there exists a sequence, $\{\phi_m\}_{m=1}^{\infty}$, of solutions to (1), (2) such that $\phi_m(x) > 0$ for $x \in (0, 1]$, and

$$\| \phi_m \| \geq \| \phi_{m+1} \| \text{ and } \lim_{m\to\infty} \| \phi_m \| = 0,$$

uniformly on $[0, 1]$.

Let

$$\overline{m} = \inf\big\{G(x, t) : (x, t) \in [p, 1] \times [p, 1]\big\} > 0.$$

From condition (ii) on f, we have that

$$\lim_{r\to 0^+} f(x, r) = \infty,$$

uniformly on compact subsets of $(0, 1]$. Hence, there exists a $\delta > 0$ such that, for $x \in [p, 1]$ and $0 < r < \delta$, we have

$$f(x, r) > \frac{1}{\overline{m}(1-p)}.$$

There exists an m_0 such that, $m \geq m_0$ implies

$$0 < \phi_m(x) < \frac{\delta}{2}, \quad x \in (0, 1].$$

So, for $x \in [p, 1]$ and $m \geq m_0$, we have

$$\begin{aligned} \phi_m(x) &\geq \overline{m} \int_p^1 f(t, \phi_m(t))dt \\ &\geq \overline{m} \int_p^1 f(t, \delta/2)dt \\ &> 1. \end{aligned}$$

This is a contradiction to the assumption that $\lim_{m\to\infty} \| \phi_m \| = 0$ uniformly on $[0, 1]$. Hence, $R \leq \| \phi \|$. □

Thus, altogether, Lemma 4 and Lemma 5 give us, for ϕ a solution of (1), (2),

$$R \leq \| \phi \| \leq S.$$

We now state our existence result.

Theorem 6. *Assume (i)-(iii) are satisfied. Then (1), (2) have a positive solution ϕ in D.*

Proof. For each m, let $\psi_m = T(m)$, where m is the constant function of that value on $[0, 1]$. In particular,

$$\psi_m(x) = \int_0^1 G(x, t)f(t, m)dt.$$

Since $f(x,r)$ is decreasing in r, and as we have observed, T is also a decreasing mapping, we have

$$0 < \psi_{m+1}(x) \leq \psi_m(x),$$

for $x \in (0,1]$. And by (ii), $\lim_{m\to\infty} \psi_m(x) = 0$ uniformly on $[0,1]$.

We now define $f_m : (0,1] \times [0,\infty) \to (0,\infty)$ as

$$f_m(x,r) \quad = \quad f(x, \max\{r, \psi_m(x)\}).$$

Then, f_m is continuous and f_m does not possess a singularity at $r = 0$. Moreover, for $(x,t) \in (0,1] \times (0,\infty)$ we have that

$$f_m(x,t) \leq f(x,t),$$

and in particular,

$$f_m(x, \phi_m(x)) = f(x, \max\{\phi_m(x), \psi_m(x)\} \leq f(x, \psi_m(x)).$$

Next, we define a sequence of operators, $T_m : \mathcal{K} \to \mathcal{K}$, for $\phi \in \mathcal{K}$ and $x \in [0,1]$, by

$$T_m\phi(x) := \int_0^1 G(x,t) f_m(t, \phi(t))dt.$$

It is standard that each T_m is a compact mapping on $\mathcal{K}$. Moreover, $T_m(0) \geq 0$ and $T_m^2(0) \geq 0$. Thus, by Theorem 1, T_m has a fixed point in $\mathcal{K}$. Thus, for all m there exists a $\phi_m \in \mathcal{K}$ such that $T_m\phi_m = \phi_m$. Hence, for $m \geq 1$, ϕ_m satisfies the boundary conditions (2).

Moreover, for each $\phi_m \in \mathcal{K}$, we note that

$$\begin{aligned}
T_m\phi_m(x) &= \int_0^1 G(x,t) f_m(t, \phi_m(t))dt \\
&= \int_0^1 G(x,t) f(t, \max\{\phi_m(t), \psi_m(t)\})dt \\
&\leq \int_0^1 G(x,t) f(t, \psi_m(t))dt \\
&= T\psi_m(x),
\end{aligned}$$

which gives us, $\phi_m(x) = T_m\phi_m(x) \leq T\psi_m(x)$ for all m.

Arguing much along the same lines of Lemma 4 and using $T_m\phi_m(x) \leq T\psi_m(x)$, it is fairly straightforward to show that there exists an $S > 0$ such that $\| \phi_m \| \leq S$, for all m. Similarly, we can follow the proof of Lemma 5 to show that there exists an $R > 0$ such that $\| \phi_m \| \geq R$. Hence, altogether we have that

$$R \leq \| \phi_m \| \leq S,$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)

for all m.

As observed before, we know that $\phi_m(x) \geq g_1(x) \parallel \phi_m \parallel \geq g_1(x)R = g_R(x)$, for $x \in [0,1]$ and for all m. This implies that the sequence $\{\phi_m\}_{m=1}^{\infty}$ is contained in the order interval $< g_R, S >$, where S is the constant function of that value on $[0,1]$. That is, the sequence $\{\phi_m\}_{m=1}^{\infty}$ is contained in D. Thus, using the fact that T is a compact mapping, we may assume that $\lim_{m\to\infty} T\phi_m$, say ϕ^* exists.

To conclude the proof of this theorem, we show that

$$\lim_{m\to\infty}(T\phi_m(x) - \phi_m(x)) = 0.$$

This will give us $\phi^* \in < g_\theta, S >$ and

$$T\phi^*(x) = T(\lim_{m\to\infty} T\phi_m(x)) = T(\lim_{m\to\infty} \phi_m(x)) = \lim_{m\to\infty} T\phi_m(x) = \phi^*(x).$$

To that end, let us set $\theta = R$. Then $g_\theta \leq \phi_m$ for all m. Let $\epsilon > 0$ be given and choose δ such that $0 < \delta < 1$ and such that

$$\int_0^\delta f(t, g_\theta(t))dt < \frac{\epsilon}{2M},$$

where $M = \max\{G(x,t) : (x,t) \in [0,1] \times [0,1]\}$. Then there exists an m_0 such that, for $m \geq m_0$,

$$\psi_m(t) \leq g_\theta(t) \leq \phi_m(t), \quad t \in [\delta, 1].$$

Thus, for $t \in [\delta, 1]$,

$$f_m(t, \phi_m(t)) = f(t, \phi_m(t))$$

and we have that, for $0 \leq x \leq 1$,

$$\begin{aligned} T\phi_m(x) - \phi_m(x) &= T\phi_m(x) - T_m\phi_m(x) \\ &= \Big[\int_0^\delta G(x,t)f(t,\phi_m(t))dt + \int_\delta^1 G(x,t)f(t,\phi_m(t))dt\Big] \\ &\quad -\Big[\int_0^\delta G(x,t)f_m(t,\phi_m(t))dt + \int_\delta^1 G(x,t)f_m(t,\phi_m(t))dt\Big] \\ &= \int_0^\delta G(x,t)f(t,\phi_m(t))dt - \int_0^\delta G(x,t)f_m(t,\phi_m(t))dt. \end{aligned}$$

Thus we have, for $0 \leq x \leq 1$,

$$\begin{aligned} \mid T\phi_m(x) - \phi_m(x) \mid &\leq M\Big[\int_0^\delta f(t,\phi_m(t))dt + \int_0^\delta f(t, \max\{\phi_m(t), \psi_m(t)\})dt\Big] \\ &\leq M\Big[\int_0^\delta f(t,\phi_m(t))dt + \int_0^\delta f(t,\phi_m(t))dt\Big] \end{aligned}$$

$$\begin{aligned} &\le 2M\int_0^\delta f(t,g_\theta(t))dt \\ &< \epsilon. \end{aligned}$$

Since $x \in [0,1]$ was arbitrary, we have for $m \ge m_0$, $\| T\phi_m - \phi_m \| \le \epsilon$, and hence the result,

$$T\phi^* = \phi^*.$$

In particular, ϕ^* is a desired positive solution of (1), (2), and the proof is complete. □

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 1, No. 1 (2010)